QUADRATURE ON THE CUBED SPHERE: THE LOW RESOLUTION CASE

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ABSTRACT. Since more than 30 years, the equiangular Cubed Sphere CS_N has been used in many domains of Computational Physics, in competition with other spherical grids (the longitude-latitude grid, the icosahedral grid, the yin-yang grid, the doubly periodic grid, and so on). Previous studies have analyzed the relation between the set of nodes CS_N and interpolation and approximation with Spherical Harmonics. An outcome has been the design of a series of quadrature rules. Here we continue our analysis of the Cubed Sphere by focusing our attention to the "low resolution" case $N \in \{1, 2, 3, 4\}$. In this case, the nodes are all located along meridians with longitude $\frac{\pi}{4}$ mod $\frac{\pi}{2N}$, and we exhibit a $4N - 1$ accurate quadrature rule with nodes CS_N and explicit positive weights. The geometry of the grid is used to compute the weights by integration of a specific Lagrange interpolating polynomial, and to prove the optimality of the rule. The particular rule obtained for $N = 4$ uses the 98 nodes of the grid CS_4, and reaches a remarkable degree of accuracy of 15. A series of numerical results are shown, assessing the interest of the present analysis.

1. INTRODUCTION

The Cubed Sphere CS_N belongs to the family of spherical grids whose nodes are clustered in six panels mirroring the six faces of a Cube [4,16,18]. Within the six panels, the nodes are arranged along great circle sections, with vertical or horizontal orientation. In [6–8], various theoretical and numerical results have been presented, supporting the interest of CS_N as a discrete spherical model, in relation to particular subsets of Spherical Harmonics (called SH hereafter). Here, we continue the study of the relation $\mathrm{CS}_N/\,\mathrm{SH}$ by restricting our attention to the particular case of "small" Cubed Sphere grids CS_N with the resolution parameter $N \in \{1, 2, 3, 4\}$. The grid CS_1 is just the 8 vertices of the inscribed cube. The case $N = 4$ corresponds to a 98 nodes grid. In these four cases, the nodes of CS_N are all located along a set of meridians with longitude $\frac{\pi}{4}$ mod $\frac{\pi}{2N}$, a property not true for $N > 4$. We take benefit from this property in a framework of quadrature rules.

We analytically integrate new Lagrange interpolating polynomials. Explicit formulas for the associated weights are derived, showing their positivity. The order of the rule is $4N - 1$, the degree of the interpolating polynomial. It is more accurate than $2N - 1$, which is the "cut-off" order naturally associated with CS_N [8]. This shows that "small" Cubed Spheres inherit some extra approximation accuracy, a mathematical observation of interest in itself. In addition, this accuracy is proved to be optimal. Of particular interest is the CS_4 grid (98 nodes), associated with a rule of order 15. This simple rule, with nodes and associated positive weights given analytically, seems new. It can be attractive to use in certain circumstances.

The outline is as follows. Section 2 fixes the geometric notation. In Section 3 our quadrature rule is described as a corollary of a new Lagrange interpolation. Section 4 comments on the relations with other quadrature rules and several numerical results are shown. The observed accuracy competes with the famous Lebedev's rules with similar spatial resolution. Some perspectives are drawn in Section 5. Finally, we provide in Appendix 6 a short Matlab code which implements the rules.

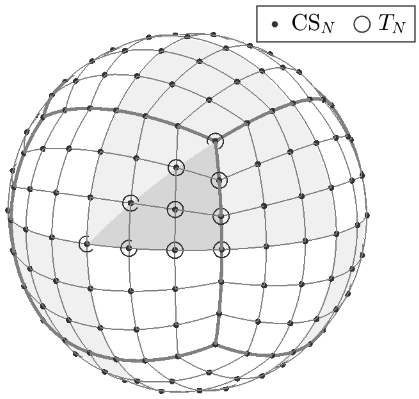

Figure 1. Equiangular Cubed Sphere CS_N ($N = 6$). By octahedral symmetry, the grid CS_N (.) can be deduced from its restriction T_N (o) to the spherical triangle $\{0 \le x_3 \le x_2 \le x_1 \le 1\}$ (in gray). Bold lines (in gray), resp. the chessboard (in white/light gray), show the radial projection of the cube $[-1, 1]^3$, resp. the dual octahedron, on the sphere $\mathbb{S}^2$. Plotted lines are great circle sections passing through points of CS_N and through vertices of the octahedron.

2. Cubed Sphere notation

For a given resolution parameter $N \ge 1$, the Cubed Sphere $\mathrm{CS}_N \subset \mathbb{S}^2$, displayed in Fig. 1, is the set of nodes given by

$$\mathrm{CS}_N \triangleq \Big\{ \tfrac{1}{\sqrt{1+u^2+v^2}}(\pm 1, u, v),\ \tfrac{1}{\sqrt{1+u^2+v^2}}(u, \pm 1, v),\ \tfrac{1}{\sqrt{1+u^2+v^2}}(u, v, \pm 1);\ u = \tan\phi_j,\ v = \tan\phi_k,\ 0 \le j, k \le N \Big\},$$

where

$$\phi_i := -\tfrac{\pi}{4} + i\tfrac{\pi}{2N} \in [-\tfrac{\pi}{4}, -\tfrac{\pi}{4} + \pi), \quad 0 \le i \le 2N - 1.$$

Let $\mathcal{G}$ be the octahedral group,

$$\mathcal{G} = \left\{ \begin{bmatrix} \epsilon_1 e_{\sigma_1} & \epsilon_2 e_{\sigma_2} & \epsilon_3 e_{\sigma_3} \end{bmatrix},\ \sigma \in \mathfrak{S}_3,\ \epsilon \in \{-1, 1\}^3 \right\}, \tag{1}$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, and $\mathfrak{S}_3$ denotes the permutation group of $\{1, 2, 3\}$. It turns out that the set of nodes CS_N is invariant under the action of the group $\mathcal{G}$, as proved in [4]. Therefore, the grid CS_N can be expressed as a disjoint union of orbits,

$$\mathrm{CS}_N = \bigcup_{z \in T_N} O(z), \quad \text{with} \quad O(z) := \{Qz,\ Q \in \mathcal{G}\}, \tag{2}$$

where $T_N \subset \mathrm{CS}_N$ is the subset of nodes located in the spherical triangle $\{x \in \mathbb{S}^2 : 0 \le x_3 \le x_2 \le x_1 \le 1\}$,

$$T_N \triangleq \left\{ z_{j,k} := \tfrac{1}{\sqrt{1+\tan^2\phi_j + \tan^2\phi_k}}(1, \tan\phi_j, \tan\phi_k), \quad \lceil \tfrac{N}{2} \rceil \le k \le j \le N \right\}. \tag{3}$$

In the particular case $N = 1, 2, 3, 4$ (low-resolution Cubed Sphere), the following geometric property holds, see Fig. 2.

Lemma 1. *For $N \in \{1, 2, 3, 4\}$, the set of nodes CS_N is included in a set of equiangular meridians, (see Fig. 2),*

$$\mathrm{CS}_N \subset \mathcal{M}_N := \{x(\theta, \phi),\ \text{with}\ -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}, \quad \phi \equiv \tfrac{\pi}{4}\ [\tfrac{\pi}{2N}]\}, \quad N \in \{1, 2, 3, 4\}, \tag{4}$$

with

$$x(\theta, \phi) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta), \quad -\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2},\ \phi \in \mathbb{R}. \tag{5}$$

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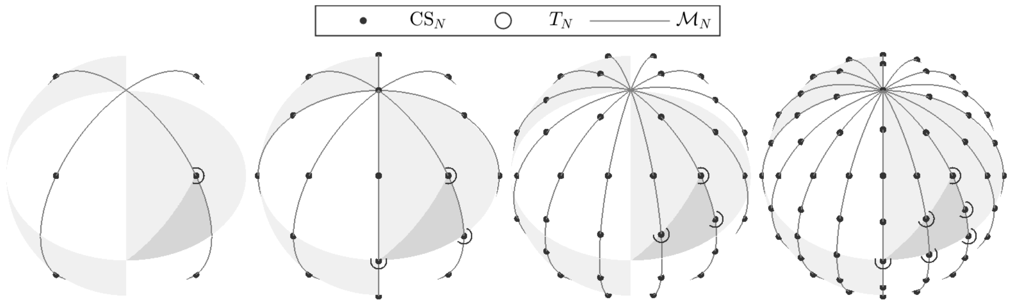

$N = 1$ $N = 2$ $N = 3$ $N = 4$

FIGURE 2. For $N \in \{1, 2, 3, 4\}$, the equiangular Cubed Sphere CS_N is included in a set $\mathcal{M}_N$ based on equiangular meridian circles, as in (4); the "generating" set T_N is reported in Table 1.

N	x_1	x_2	x_3	$\omega_{\text{opt}}(x_1, x_2, x_3)$	$\lvert\mathrm{CS}_N\rvert$	Degree
1	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{\pi}{2}$	8	3
2	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{9\pi}{70}$	26	7
	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{16\pi}{105}$		
	1	0	0	$\frac{4\pi}{21}$		
3	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{9\pi}{140}$	56	11
	$\frac{1}{\sqrt{2+t^2}}$	$\frac{1}{\sqrt{2+t^2}}$	$\frac{t}{\sqrt{2+t^2}}$	$\frac{61\pi}{840} - \frac{3\pi\sqrt{3}}{560}$		
	$\frac{1}{\sqrt{1+2t^2}}$	$\frac{t}{\sqrt{1+2t^2}}$	$\frac{t}{\sqrt{1+2t^2}}$	$\frac{61\pi}{840} + \frac{3\pi\sqrt{3}}{560}$		
				with $t = 2 - \sqrt{3}$		
4	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{729\pi}{20020}$	98	15
	$\frac{1}{\sqrt{2+s^2}}$	$\frac{1}{\sqrt{2+s^2}}$	$\frac{s}{\sqrt{2+s^2}}$	$\frac{2053\pi}{51480} - \frac{183\pi\sqrt{2}}{80080}$		
	$\frac{1}{\sqrt{1+2s^2}}$	$\frac{s}{\sqrt{1+2s^2}}$	$\frac{s}{\sqrt{1+2s^2}}$	$\frac{2053\pi}{51480} + \frac{183\pi\sqrt{2}}{80080}$		
	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$\frac{512\pi}{15015}$		
	$\frac{1}{\sqrt{1+s^2}}$	$\frac{s}{\sqrt{1+s^2}}$	0	$\frac{2048\pi}{45045}$		
	1	0	0	$\frac{736\pi}{15015}$		
				with $s = \sqrt{2} - 1$		

TABLE 1. Optimal quadrature weights on the Cubed Sphere CS_N, $1 \le N \le 4$. Weights $\omega_{\text{opt}}(x_1, x_2, x_3)$, $(x_1, x_2, x_3) \in T_N$, with T_N defined in (3), are enumerated. An octahedral weight $\omega_{\text{opt}} : \mathrm{CS}_N \to \mathbb{R}$, with $|\mathrm{CS}_N| = 6N^2 + 2$ nodes, is deduced by octahedral invariance. Theorem 3 shows that the corresponding quadrature rule has the maximum degree of accuracy, $4N - 1$.

3. QUADRATURE RULE ON THE CUBED SPHERE

3.1. **A $4N - 1$ Lagrange polynomial on CS_N.** For all $D \ge 0$, the space of polynomials in $(x_1, x_2, x_3) \in \mathbb{R}^3$ with total degree less or equal to D is denoted by $\mathcal{P}_D$,

$$\mathcal{P}_D = \mathrm{span}\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto x_1^\alpha x_2^\beta x_3^\gamma, \text{ with } 0 \le \alpha, \beta, \gamma \le D,\ \alpha + \beta + \gamma \le D\}.$$

The space of Spherical Harmonics of degree less or equal to D, denoted by $\mathcal{Y}_D$, is defined by restricting the harmonic polynomials in $\mathcal{P}_D$ to the sphere $\mathbb{S}^2$. A basic result is that [2, Corollary 2.15]

$$\mathcal{Y}_D = \{p|_{\mathbb{S}^2},\ p \in \mathcal{P}_D\}.$$

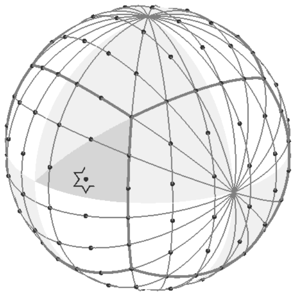

FIGURE 3. Covering of the equiangular Cubed Sphere CS_N by means of great circles. Here, for $z \in T_N$, we have plotted $2N-1$ great circles through the poles $(0,0,\pm1)$, and $2N-1$ great circles through $(0,\pm1,0)$. These circles cover $\mathrm{CS}_N \setminus \{z,-z\}$. By construction, the polynomial L_z in (8) vanishes along these circles, thus L_z vanishes on $\mathrm{CS}_N \setminus \{z,-z\}$.

We call "grid function" a real function defined at the nodes in CS_N. For a given function $f : \mathbb{S}^2 \to \mathbb{R}$, we call f^* the grid function $f|_{\mathrm{CS}_N}$. Studying CS_N as collocation nodes for $p \in \mathcal{P}_D$, or equivalently for $p \in \mathcal{Y}_D$, naturally leads to analyze the application

$$p \in \mathcal{P}_D \mapsto p^* \triangleq p|_{\mathrm{CS}_N}. \tag{6}$$

An important tool for analyzing the surjectivity of the application (6) is the construction of suitable Lagrange interpolating polynomials. For all $z \in \mathrm{CS}_N$, an elementary Lagrange polynomial L_z satisfies

$$L_z(z') = \begin{cases} 1, & \text{if } z' = z, \\ 0, & \text{if } z' \neq z, \end{cases} \quad z, z' \in \mathrm{CS}_N. \tag{7}$$

With L_z at hand, the polynomial $p = \sum_{z \in \mathrm{CS}_N} f(z) L_z \in \mathcal{P}_D$ interpolates f^* in $\mathcal{P}_D$.

A straightforward elementary polynomial L_z is obtained by

$$L_z(x) = \prod_{z' \in \mathrm{CS}_N \setminus \{z\}} \frac{1 - z' \cdot x}{1 - z' \cdot z}, \quad z \in \mathrm{CS}_N.$$

This shows the surjectivity of the application (6) for D large enough ($D = |\mathrm{CS}_N| - 1 = 6N^2 + 1$). In fact, using the great circle arrangement structure of CS_N, a more useful Lagrange polynomial of degree $4N-1$ can be constructed, [5, Lemma 6.1]. Basically, instead of considering one tangent plane per grid point ($1 - z' \cdot x = 0$) as above, we introduce great circles passing through "many" points of the grid ($u \cdot x = 0$).

Theorem 2 (Lagrange interpolation with degree $4N-1$ on CS_N)**.** *Let $f : \mathbb{S}^2 \to \mathbb{R}$ be a regular function. Then, for all $N \geq 1$, there exists a polynomial $p \in \mathcal{P}_{4N-1}$ interpolating f^* at CS_N nodes,* i.e. *$p^* = f^*$.*

Proof. We build an elementary polynomial L_z for all $z \in \mathrm{CS}_N$. Consider first the case $z = z_{j,k} \in T_N$ defined in (3), with $\lceil \frac{N}{2} \rceil \leq k \leq j \leq N$. The polynomial $L_z \in \mathcal{P}_{4N-1}$ is defined as a product of $4N-1$ polynomials of degree 1 by

$$L_z(x) = \frac{1 + z \cdot x}{2} \left(\prod_{\substack{0 \leq m \leq 2N-1 \\ m \neq j}} \frac{(-\sin\phi_m, \cos\phi_m, 0) \cdot x}{(-\sin\phi_m, \cos\phi_m, 0) \cdot z} \right) \left(\prod_{\substack{0 \leq n \leq 2N-1 \\ n \neq k}} \frac{(-\sin\phi_n, 0, \cos\phi_n) \cdot x}{(-\sin\phi_n, 0, \cos\phi_n) \cdot z} \right). \tag{8}$$

By construction, L_z vanishes on the tangent plane at $-z$ ($1 + z \cdot x = 0$), yielding $L_z(-z) = 0$. Also, L_z vanishes on great circles not containing z, defined by $(-\sin\phi_m, \cos\phi_m, 0) \cdot x = 0$ with $m \neq j$, $(-\sin\phi_n, 0, \cos\phi_n) \cdot x = 0$ with $n \neq k$. As shown in Fig. 3, each node $z' \in \mathrm{CS}_N \setminus \{z, -z\}$ belongs to at least one of these circles, yielding $L_z(z') = 0$. Furthermore, the factors are normalized such that $L_z(z) = 1$. Finally, L_z satisfies (7). Next, fix a node $z \in \mathrm{CS}_N \setminus T_N$. In this case, there are a node $z_{j,k} \in T_N$ (with $\lceil \frac{N}{2} \rceil \leq k \leq j \leq N$), and an orthogonal matrix $Q \in \mathcal{G}$, such that $z = Qz_{j,k}$. Then, the polynomial defined by $L_z(x) \triangleq L_{z_{j,k}}(Q^\intercal x)$, where $L_{z_{j,k}}$ as in (8) satisfies (7). □

3.2. **Optimal quadrature rule.** Let f^* be the grid function corresponding to a given function f. For $N \in \{1, 2, 3, 4\}$, consider a quadrature rule (Q) with nodes in CS_N,

$$(Q) \qquad \int_{\mathbb{S}^2} f(x) \mathrm{d}\sigma(x) \simeq \sum_{x \in \mathrm{CS}_N} \omega(x) f^*(x). \tag{9}$$

A particular class of rule (Q) deals with "octahedral", in the sense that they are deduced by octahedral symmetry from weights specified at the nodes $x \in T_N$ only. The following theorem holds.

Theorem 3 (Optimal quadrature rule on low-resolution Cubed Spheres)**.** *Let $N \in \{1, 2, 3, 4\}$. (i) The octahedral weight $\omega = \omega_{\mathrm{opt}} : \mathrm{CS}_N \to (0, \infty)$ given in Table 1 defines a quadrature rule (Q_{opt}) with degree of accuracy $4N - 1$,* i.e.

$$\forall p \in \mathcal{P}_{4N-1}, \quad \textstyle\int_{\mathbb{S}^2} p(x)\, \mathrm{d}\sigma = \sum_{x \in \mathrm{CS}_N} \omega(x)\, p(x), \tag{10}$$

$$\exists p \in \mathcal{P}_{4N}, \quad \textstyle\int_{\mathbb{S}^2} p(x)\, \mathrm{d}\sigma \neq \sum_{x \in \mathrm{CS}_N} \omega(x)\, p(x). \tag{11}$$

(ii) The rule (Q_{opt}) is the optimal one in the following sense. Any quadrature rule (Q) on CS_N with a weight function $\omega \neq \omega_{\mathrm{opt}}$, is less accurate than Q_{opt}, i.e.

$\forall \omega : \mathrm{CS}_N \to \mathbb{R}$,

$$(\exists x \in \mathrm{CS}_N, \omega(x) \neq \omega_{\mathrm{opt}}(x)) \Rightarrow \left(\exists p \in \mathcal{P}_{4N-1}, \textstyle\int_{\mathbb{S}^2} p(x)\, \mathrm{d}\sigma \neq \sum_{x \in \mathrm{CS}_N} \omega(x)\, p(x)\right). \tag{12}$$

Proof. First, assume that $\omega : \mathrm{CS}_N \to \mathbb{R}$ is a weight grid function associated with a quadrature rule satisfying (10) (exact on $\mathcal{P}_{4N-1}$). We show that, necessarily, ω does possess the octahedral symmetry and takes the values given in Table 1, fourth column. Fix $z \in \mathrm{CS}_N$, and apply the quadrature rule for an elementary Lagrange polynomial $L_z \in \mathcal{P}_{4N-1}$ in (7). We obtain

$$\omega(z) = \int_{\mathbb{S}^2} L_z(x)\, \mathrm{d}\sigma.$$

Next, for all $Q \in \mathcal{G}$, applying the quadrature rule to $L_z(Q^\intercal \cdot)$ yields

$$\omega(Qz) = \int_{\mathbb{S}^2} L_z(Q^\intercal x)\, \mathrm{d}\sigma = \int_{\mathbb{S}^2} L_z(y)\, \mathrm{d}\sigma = \omega(z), \quad z \in \mathrm{CS}_N,\ Q \in \mathcal{G}.$$

This proves the octahedral invariance of ω. Assume now (without restriction) that $z = z_{j,k}$ with $\lceil \frac{N}{2} \rceil \leq k \leq j \leq N$, and consider the polynomial L_z in (8). We evaluate the integral of L_z in (8) in the coordinate system (5),

$$\omega(z) = \int_{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]} \int_{\phi \in [-\frac{\pi}{4}, \frac{7\pi}{4}]} L_z(x(\theta, \phi)) \mathrm{d}\phi \cos\theta \mathrm{d}\theta.$$

The inner integrand is a trigonometric polynomial in ϕ with degree smaller or equal to $4N - 1$. Therefore, it is exactly evaluated by the trapezoidal rule with step $\frac{\pi}{2N}$. Moreover, $L_z(x(\theta, \phi)) = 0$ if $\phi \equiv \phi_m\, [\pi]$ with $m \neq j$, since L_z vanishes on the meridian circle with longitude $\phi = \phi_m$. Therefore,

$$\omega(z_{j,k}) = \frac{\pi}{2N} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [L_{z_{j,k}}(x(\theta, \phi_j)) + L_{z_{j,k}}(x(\theta, \phi_j + \pi))] \cos\theta \mathrm{d}\theta.$$

Again, the integrand is a trigonometric polynomial (in θ), with degree at most $4N$. Calculating this integral has been performed using symbolic computation. The values are reported in Table 1, fourth column.

Conversely, consider the octahedral weight ω in Table 1. The reported values are $\omega(z)$ with $z \in T_N$; they are extended to CS_N using octahedral invariance: for every $z' \in \mathrm{CS}_N$, $\omega(z') := \omega(z)$, where $z \in T_N$ is the unique point in T_N such that $z' \in O(z)$. We show the exactness property (10). The number of properties to verify is reduced according to [13, 17]. Indeed, as a consequence

N	Polynomials $v_1^\alpha v_2^\beta$ with degree $4\alpha+6\beta \le 4N-1$
1	1
2	1, v_1, v_2
3	1, v_1, v_2, v_1^2, v_1v_2
4	1, v_1, v_2, v_1^2, v_1v_2, v_1^3, v_2^2, $v_1^2v_2$

Table 2. Polynomials to be considered to insure the exactness of an octahedral quadrature rule in $\mathcal{P}_{4N-1}$ (v_1 and v_2 are given in (13)). See [13].

of octahedral symmetry, it is (necessary and) sufficient to show the exactness property only for those polynomials which are invariant by $\mathcal{G}$ [17]. Moreover, any polynomial invariant by $\mathcal{G}$ coincides on the sphere with a polynomial in the variables v_1, v_2 given by [13],

$$v_1 = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, \quad v_2 = x_1^2x_2^2x_3^2. \tag{13}$$

Therefore, it is sufficient to verify that $\int_{\mathbb{S}^2} p(x)\mathrm{d}\sigma = \sum_{x\in \mathrm{CS}_N} \omega(x)p(x)$ for the list of polynomials p in Table 2. This verification has been performed using symbolic computation. We have proved so far that the octahedral weight ω from Table 1 is the only grid function on CS_N guarantying the exactness property (10). To conclude the proof, we show that the degree of accuracy is exactly $4N-1$, using a counterexample of degree $4N$ which proves (11). We use the Spherical Harmonics $q = Y_{2N}^{-2N}(x(\theta, \phi - \frac{\pi}{4})) \in \mathcal{Y}_{2N}$, introduced in [8]. Up to a constant factor, q is given by $(\cos\theta)^{2N} \sin[2N(\phi - \frac{\pi}{4})]$; therefore, $q(x) = 0, x \in \mathrm{CS}_N$, due to (4). Consider now $p \in \mathcal{P}_{4N}$ such that $p|_{\mathbb{S}^2} = q^2 \in \mathcal{Y}_{4N}$. Since $p(x) = 0$, $x \in \mathrm{CS}_N$, we have $\sum_{x\in\mathrm{CS}_N} \omega(x)p(x) = 0$, whereas q is unitary in $L^2(\mathbb{S}^2)$, so $\int_{\mathbb{S}^2} p(x)\mathrm{d}\sigma = 1$. □

Remark 4. For $N = 1, 2$, the number of polynomials to be considered coincides with the number of weights ($|T_N|$). However, in the case $N = 3$ (resp. $N = 4$), there are two additional polynomials in the list in Table (1), namely 5 polynomials (resp. 8 polynomials) compared to $|T_N| = 3$ weights, (resp. $|T_N| = 6$ weights). It is therefore remarkable to reach the degree of accuracy $4N-1$ for $N = 3$ and $N = 4$.

3.3. **Least squares on CS_N with Spherical Harmonics.** In [8], a study of the least square approximation at the nodes of CS_N with the space $\mathcal{Y}_{2N-1}$ has been introduced in the general case $N \ge 1$. The treatment of the particular case $N \in \{1, 2, 3, 4\}$ can be completed as follows.

Corollary 5 (Least squares spherical harmonics)**.** *Let $N \in \{1, 2, 3, 4\}$, and let ω be the octahedral weight in Table 1. The following claims hold true.*
(i) The Spherical Harmonics $(Y_n^m)^*, |m| \le n \le 2N-1$, define an orthonormal basis of the space $\{y^*,\, y \in \mathcal{Y}_{2N-1}\}$ for the discrete inner product

$$\langle f, g\rangle = \sum_{x\in\mathrm{CS}_N} \omega(x)f(x)g(x), \quad f, g : \mathrm{CS}_N \to \mathbb{R}. \tag{14}$$

(ii) The linear mapping $p \in \mathcal{Y}_{2N-1} \mapsto p^*$ is injective.
(iii) For every $f : \mathrm{CS}_N \to \mathbb{R}$, the weighted least squares problem

$$\inf_{p\in\mathcal{Y}_{2N-1}} \sum_{x\in\mathrm{CS}_N} \omega(x)(p(x) - f(x))^2 \tag{WLS}$$

has a unique solution, and it is given by

$$p = \sum_{|m|\le n\le 2N-1} \hat{p}_n^m Y_n^m, \quad \textit{with} \quad \hat{p}_n^m = \sum_{x\in\mathrm{CS}_N} \omega(x)f(x)Y_n^m(x).$$

Proof. (i) The symmetric bilinear mapping (14) defines an inner product because ω is positive. For all $p, q \in \mathcal{Y}_{2N-1}$, we have $pq \in \mathcal{Y}_{4N-2}$, so (10) implies

$$\langle p, q\rangle_{L^2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} pq\,\mathrm{d}\sigma = \sum_{x\in\mathrm{CS}_N} \omega(x)p(x)q(x) = \langle p^*, q^*\rangle. \tag{15}$$

The spherical harmonics Y_n^m, $|m| \le n \le 2N-1$, define an orthonormal basis of $\mathcal{Y}_{2N-1}$, for $\langle \cdot, \cdot \rangle_{L^2(\mathbb{S}^2)}$. The relation above implies that $(Y_n^m)^*$, $|m| \le n \le 2N-1$, define an orthonormal basis of $\{y^*,\, y \in \mathcal{Y}_{2N-1}\}$, for $\langle \cdot, \cdot \rangle$.

(ii) Let $p = \sum_{|m| \le n \le 2N-1} \hat{p}_n^m Y_n^m \in \mathcal{Y}_{2N-1}$. We obtain the expansion of p^* in the orthonormal basis $(Y_n^m)^*$, $|m| \le n \le 2N-1$, by restriction to CS_N: $p^* = \sum_{|m| \le n \le 2N-1} \hat{p}_n^m (Y_n^m)^*$. This implies $\hat{p}_n^m = \langle p^*, (Y_n^m)^* \rangle$, $|m| \le n \le 2N-1$, which proves the injectivity[1] of $p \in \mathcal{Y}_{2N-1} \mapsto p^*$.

(iii) Let $f : \mathrm{CS}_N \to \mathbb{R}$ and $p = \sum_{|m| \le n \le 2N-1} \hat{p}_n^m Y_n^m \in \mathcal{Y}_{2N-1}$. The cost in (WLS) is nothing else but the squared norm $\langle p^* - f, p^* - f \rangle$. This quantity is minimal if, and only if, p^* is the orthogonal projection of f on $\{y^*,\, y \in \mathcal{Y}_{2N-1}\}$ for $\langle \cdot, \cdot \rangle$. In the orthonormal basis $(Y_n^m)^*$, $|m| \le n \le 2N-1$, this is equivalent to $\hat{p}_n^m = \langle f, (Y_n^m)^* \rangle = \sum_{x \in \mathrm{CS}_N} \omega(x) f(x) Y_n^m(x)$. □

4. Applications and comments

4.1. **Case $N = 1$.** In the case $N = 1$, the weight $\omega(x) = \frac{\pi}{2}$ is uniform. It simply corresponds to the area of a face of the octahedron projected on the sphere. It can be related to a Gaussian quadrature rule as follows. The grid CS_1 coincides with a longitude-latitude grid, with 4 subdivisions in $\phi \in [-\frac{\pi}{4}, \frac{7\pi}{4}]$, and 2 Gauss-Legendre nodes in $x_3 \in [-1, 1]$:

$$\mathrm{CS}_1 = \left\{ (\cos\phi\,(1-x_3^2)^{1/2}, \sin\phi\,(1-x_3^2)^{1/2}, z), \quad (\phi, x_3) \in \{\phi_i,\, 0 \le i \le 3\} \times \{-\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\} \right\}.$$

The uniform weight $\omega = \frac{\pi}{2}$ on CS_1 can be deduced from a trapezoidal rule in ϕ and a Gauss-Legendre rule in x_3:

$$\int_{\mathbb{S}^2} f(x_1, x_2, x_3)\, \mathrm{d}\sigma = \int_{-1}^{1} \int_{-\frac{\pi}{4}}^{\frac{7\pi}{4}} f(\cos\phi\,(1-x_3^2)^{1/2}, \sin\phi\,(1-x_3^2)^{1/2}, x_3)\, \mathrm{d}\phi\, \mathrm{d}x_3$$
$$\simeq \frac{\pi}{2} \sum_{x_3 = \pm\frac{1}{\sqrt{3}}} \sum_{i=0}^{3} f(\cos\phi_i\,(1-x_3^2)^{1/2}, \sin\phi_i\,(1-x_3^2)^{1/2}, x_3) = \frac{\pi}{2} \sum_{x \in \mathrm{CS}_1} f(x).$$

This procedure is standard, and the corresponding degree of accuracy (3) is expected; see [2, Theorem 5.4].

4.2. **Comparison with Lebedev's octahedral quadrature rules.** In Table 4, we have reported the degree of accuracy of the first Lebedev's rules, versus the number of grid nodes, according to [9,12]. For a given degree of accuracy, the Lebedev's octahedral grid, with the corresponding octahedral weight, are designed to integrate polynomials up to the given degree [12]. The shape of the octahedral grid, including the number of grid nodes, is designed such that the number of unknowns coincides with the number of equations to be imposed, after reduction based on octahedral symmetry. In general, a nonlinear system of equations must be solved to obtain the grid. We refer to [12] for tables with degree between 9 and 17, and to [9,14] for source codes and numerical tables with degree between 3 and 131. Here, our approach is different: the grid is fixed from the beginning (the Cubed Sphere), and the quadrature weights are directly calculated , by integration of suitable trigonometric polynomials. Octahedral invariance and degree of accuracy are verified. Therefore, for a given degree of accuracy, the number of nodes of the Lebedev's grid may be expected to be smaller than the one of the Cubed Sphere. A comparison of Table 4 and Table 1 confirms this fact for the degrees 3, 11, 15, but indicates that the number of grid points remains close from each other. Furthermore, for the degree 7, the two grids have 26 points. After inspection of the two rules, it appears that our rule on CS_2 numerically coincides with the Lebedev's rule of degree 7[2] (it coincides with the one from [9], up to rounding errors).

[1] This corrects an argument in [8] where it was erroneoulsy claimed that, for $N \in \{3, 4\}$, any meridian circle $\phi \equiv \phi_i\, [\pi]$ contains $4N$ points from CS_N.

[2] We did not succeed in finding a reference with the analytical formula for the Lebedev's grid with degree 7.

i	$f_i(x,y,z)$	$I_i = \int_{\mathbb{S}^2} f_i(x,y,z)\,\mathrm{d}\sigma$	Ref.
1	$\exp(x)$	$14.7680137457653\cdots$	[2,6,10]
2	$\frac{3}{4}\exp[-\frac{(9x-2)^2}{4}-\frac{(9y-2)^2}{4}-\frac{(9z-2)^2}{4}]$ $+\frac{3}{4}\exp[-\frac{(9x+1)^2}{49}-\frac{9y+1}{10}-\frac{9z+1}{10}]$ $+\frac{1}{2}\exp[-\frac{(9x-7)^2}{4}-\frac{(9y-3)^2}{4}-\frac{(9z-5)^2}{4}]$ $-\frac{1}{5}\exp[-(9x-4)^2-(9y-7)^2-(9z-5)^2]$	$6.6961822200736179523\cdots$	[1,3,6,11,15]
3	$\cos(3\arccos z)\mathbf{1}(3\arccos z \le \frac{\pi}{2})$	$\frac{\pi}{8}$	[6]
4	$\mathbf{1}(z \ge \frac{1}{2})$	π	[6]

TABLE 3. Test functions and exact integration values.

4.3. **Comparison with other interpolatory rules on the Cubed Sphere.** In [6], a family of quadrature rules based on the Cubed Sphere has been introduced. The approach is based on Lagrange interpolation in a particular space of Spherical Harmonics. The considered space has been designed to guaranty existence and uniqueness of an interpolating function, as in [7]. Practically, this space is identified as a subspace of $\mathcal{Y}_{3N}$ using numerical linear algebra. The corresponding quadrature weights, computed in double precision, are available at the url indicated in [6].

The numerical results in [6, Table 7] show that the numerical degree of accuracy is $4N-1$ for $1 \le N \le 4$. This is surprising because the corresponding quadrature rule has been designed by interpolation in $\mathcal{Y}_{3N} \subset \mathcal{Y}_{4N-1}$. Also, these numerical observations, combined with Theorem 3.(ii), suggest that the numerical weights from [6] should coincide with the analytical weights from Table 1, up to rounding errors.

To assess that latter point, the relative difference between the weights from [6], and the weights from Table 1 has been computed in double precision. The obtained maximum relative difference over all weights has been found around $1.9\cdot 10^{-15}$. This assesses that with the numerical approach in [6,7], one actually evaluates the optimal quadrature weights in Table 1 (up to rounding errors).

4.4. **Integration errors of test functions.** We evaluate the integrals in Table 3, using the rule from Table 1, and Lebedev's rules in [9,12]. The series of considered functions is representative of various smoothness properties, ranging from very smooth (f_1), to discontinuous (f_4), and has already been used to test quadrature rules (references in the fourth column in Table 3). The computation has been performed in double precision.

Table 4, reports the relative errors

$$\eta_{\mathrm{L}} = |I_{\mathrm{L}} - I_i|/I_i, \quad \eta_{\mathrm{CS}} = |I_{\mathrm{CS}} - I_i|/I_i, \quad 1 \le i \le 4, \tag{16}$$

corresponding to the integration of f_i by Lebedev's rule, resp. the rule in Table 1. Moreover, we compute similar errors after several random orthogonal transformations of the grids, giving an error independent of the axes position. For each matrix $Q \in \mathbb{R}^{3\times 3}$ in a set of 1000 random orthogonal matrices, we compute the observed relative errors

$$\eta_{\mathrm{L}}(Q) = |I_{\mathrm{L}}(Q) - I_i|/I_i, \quad \eta_{\mathrm{CS}}(Q) = |I_{\mathrm{CS}}(Q) - I_i|/I_i, \quad 1 \le i \le 4, \tag{17}$$

corresponding to the integration of the "rotated" function $f_i(Q\cdot)$. The maximum errors have been reported in Table 4, and displayed in Fig. 4. As a result, the weight from Table 1 permits to compute the integrals from Table 3, with an observed accuracy that is relatively close to the one of Lebedev's rule.

5. SUMMARY

In this paper, we provide a series of analytical formulas for the optimal quadrature rule on low-resolution Cubed Spheres, with respect to the degree of accuracy. The rule uses the $6N^2+2$ nodes of CS_N. The degree of accuracy is $4N-1$, $1 \le N \le 4$. A direct computation, based on the specific geometrical structure of the considered grids. Despite the simplicity of the approach, the obtained rules are quite close to the usual rules such as the Lebedev's ones. In addition, we

Degree of accuracy	3	5	7	9	11	13	15	17
Size of the Lebedev's grid	6	14	26	38	50	74	86	110
Size of the Cubed Sphere	8		26		56		98	
$\lvert I_{\mathrm{L}} - I_1 \rvert / I_1$	5.0e-03	1.6e-05	1.4e-07	2.3e-10	1.1e-13	5.7e-15	2.5e-14	2.8e-15
$\lvert I_{\mathrm{CS}} - I_1 \rvert / I_1$	3.3e-03		1.4e-07		5.7e-13		6.0e-16	
$\max_Q \lvert I_{\mathrm{L}}(Q) - I_1 \rvert / I_1$	5.0e-03	2.7e-05	1.4e-07	3.5e-10	6.5e-13	5.8e-15	2.5e-14	2.8e-15
$\max_Q \lvert I_{\mathrm{CS}}(Q) - I_1 \rvert / I_1$	3.3e-03		1.4e-07		5.7e-13		8.4e-16	
$\lvert I_{\mathrm{L}} - I_2 \rvert / I_2$	1.6e-01	1.2e-02	2.2e-03	4.6e-03	3.1e-03	2.3e-03	1.2e-04	2.6e-04
$\lvert I_{\mathrm{CS}} - I_2 \rvert / I_2$	1.2e-01		2.2e-03		1.4e-03		3.3e-04	
$\max_Q \lvert I_{\mathrm{L}}(Q) - I_2 \rvert / I_2$	1.8e-01	7.3e-02	3.4e-02	1.8e-02	1.1e-02	1.1e-02	4.1e-03	2.6e-03
$\max_Q \lvert I_{\mathrm{CS}}(Q) - I_2 \rvert / I_2$	1.5e-01		3.4e-02		8.2e-03		3.4e-03	
$\lvert I_{\mathrm{L}} - I_3 \rvert / I_3$	4.3e+00	1.1e+00	5.2e-01	1.9e-01	4.3e-02	2.0e-01	1.4e-02	4.9e-02
$\lvert I_{\mathrm{CS}} - I_3 \rvert / I_3$	1.0e+00		5.2e-01		2.2e-01		5.1e-02	
$\max_Q \lvert I_{\mathrm{L}}(Q) - I_3 \rvert / I_3$	4.3e+00	1.4e+00	7.2e-01	2.1e-01	2.8e-01	2.7e-01	8.4e-02	9.5e-02
$\max_Q \lvert I_{\mathrm{CS}}(Q) - I_3 \rvert / I_3$	3.0e+00		7.2e-01		2.5e-01		1.1e-01	
$\lvert I_{\mathrm{L}} - I_4 \rvert / I_4$	3.3e-01	4.7e-01	3.1e-01	9.5e-03	7.2e-02	5.2e-01	1.6e-02	7.0e-02
$\lvert I_{\mathrm{CS}} - I_4 \rvert / I_4$	1.0e+00		3.1e-01		9.1e-02		2.1e-02	
$\max_Q \lvert I_{\mathrm{L}}(Q) - I_4 \rvert / I_4$	1.0e+00	4.7e-01	3.1e-01	2.5e-01	2.3e-01	5.2e-01	1.6e-01	1.2e-01
$\max_Q \lvert I_{\mathrm{CS}}(Q) - I_4 \rvert / I_4$	1.0e+00		3.1e-01		1.8e-01		1.4e-01	

TABLE 4. Accuracy of the rule from Table 1, compared with the accuracy of the Lebedev's rule from [9, 12]. In the top rows, the grid sizes are compared, versus the degree of accuracy. The next rows indicate the relative quadrature errors $|I - I_i|/I_i$ for the test functions f_i from Table 3; $I_i = \int f_i$ denotes the exact value, $I = I_{\mathrm{L}}$ is computed using the Lebedev's rule, $I = I_{\mathrm{CS}}$ is computed using Table 1. The maximum relative errors, over 1000 random orthogonal transformations of the grids are also reported; the matrix $Q \in \mathbb{R}^{3\times 3}$ browses a set of 1000 random orthogonal matrices, and $I_{\mathrm{L}}(Q)$, $I_{\mathrm{CS}}(Q)$, correspond to the quadrature rules applied on the "rotated" functions $f(Q\cdot)$.

have noticed a surprising extra accuracy in the case $N = 3$, 4, since in this case, the number of relations (or equations) satisfied by the weights is larger than the number of weights.

A simple open question (for $N \in \{2, 3, 4\}$) is to exhibit a polygonal tiling of the sphere with polygons areas given by the obtained weights; the goal being to find a mesh as "simple" as possible. Answering such a question may require to solve systems of non-linear equations.

Another still open question concerns quadrature rules on the grid CS_N, $N \geq 5$: what is the optimal quadrature rule, and what is its degree of accuracy ? Unfortunately, we cannot directly extend the method of the present work, since in that case, CS_N is no longer included in the set $\mathcal{M}_N$ of meridian circles. A hope is that a natural tiling of CS_N, $N \leq 4$, based on the optimal weights, suggests a tiling principle of CS_N, $N \geq 5$, whose areas give an accurate rule.

6. Appendix

We provide a Matlab code which implements the octahedral quadrature rules from Table 1,

$$\int_{\mathbb{S}^2} f \, \mathrm{d}\sigma \simeq \sum_{(x,y,z)\in T_N} \omega_{\mathrm{opt}}(x,y,z) \sum_{(x',y',z')\in O(x,y,z)} f(x',y',z').$$

The code measures also the accuracy, with f a rotated version of f_1 from Table 3.

```
%%%%function and its integral
f=@(x,y,z)exp(1/sqrt(14)*(x+2*y+3*z));%rotation of (x,y,z)->exp(x)
Iexact=2*pi*(exp(1)-exp(-1));%exact value of the integral int_{S^2} f ds
```

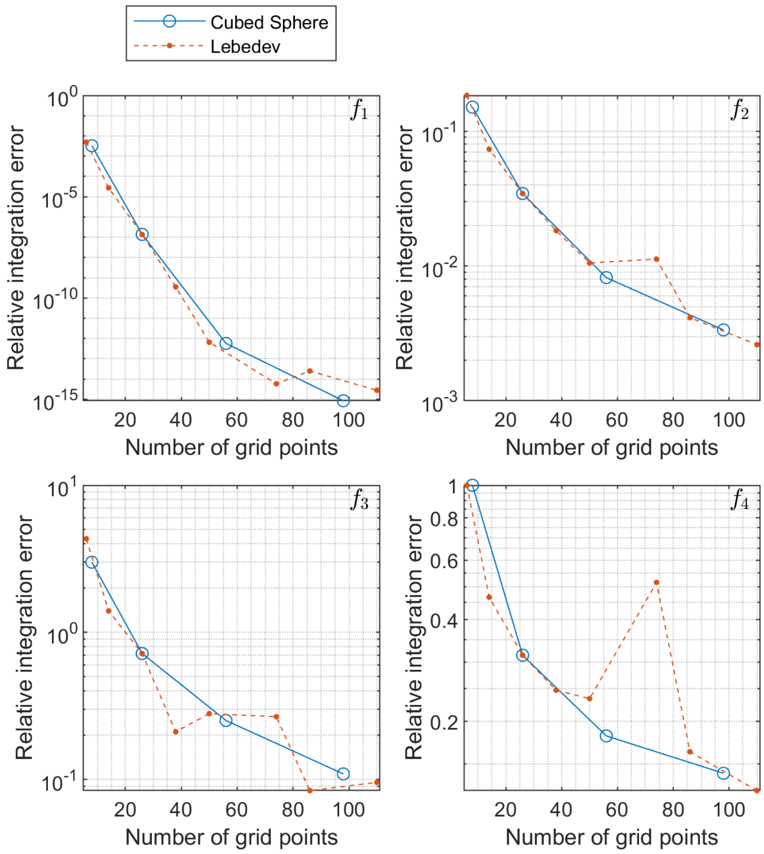

FIGURE 4. Accuracy of the rule from Table 1, compared with the accuracy of the Lebedev's rule from [9, 12], for the test functions from Table 3. The displayed errors correspond to the maximum relative quadrature errors over 1000 random orthogonal transformations of the grids, from Table 4.

```
%%%%Optimal quadrature rule on CS1, and evaluation of the relative error
N=1
u=1/sqrt(3);w=pi/2;%vertex of the cube
I=w*(f(u,u,u)+f(-u,-u,-u)+...
    +f(-u,u,u)+f(u,-u,u)+f(u,u,-u)+f(u,-u,-u)+f(-u,u,-u)+f(-u,-u,u))
(I-Iexact)/Iexact   %result: 8.2233e-04
```

```
%%%%Optimal quadrature rule on CS2, and evaluation of the relative error
N=2
u1=1/sqrt(3);w1=9*pi/70;%vertex of the cube
u2=1/sqrt(2);w2=16*pi/105;%center of an edge
w3=4*pi/21;%face center
I=w1*(f(u1,u1,u1)+f(-u1,-u1,-u1)+f(-u1,u1,u1)+f(u1,-u1,u1)...
      +f(u1,u1,-u1)+f(u1,-u1,-u1)+f(-u1,u1,-u1)+f(-u1,-u1,u1))...
 +w2*(f(u2,u2,0)+f(u2,0,u2)+f(0,u2,u2)+f(-u2,u2,0)...
      +f(-u2,0,u2)+f(0,-u2,u2)+f(u2,-u2,0)+f(u2,0,-u2)...
      +f(0,u2,-u2)+f(-u2,-u2,0)+f(-u2,0,-u2)+f(0,-u2,-u2))...
 +w3*(f(1,0,0)+f(0,1,0)+f(0,0,1)+f(-1,0,0)+f(0,-1,0)+f(0,0,-1))
(I-Iexact)/Iexact   %result: -1.6486e-08
```

```
%%%%Optimal quadrature rule on CS3, and evaluation of the relative error
N=3
u1=1/sqrt(3);w1=9*pi/140;%vertex of the cube
t=2-sqrt(3);
u4=1/sqrt(2+t^2);v4=t*u4;w4=61*pi/840-3*pi*sqrt(3)/560;%edge of the cube
u5=1/sqrt(1+2*t^2);v5=t*u5;w5=61*pi/840+3*pi*sqrt(3)/560;%diagonal of a face
I=w1*(f(u1,u1,u1)+f(-u1,-u1,-u1)+f(-u1,u1,u1)+f(u1,-u1,u1)...
      +f(u1,u1,-u1)+f(u1,-u1,-u1)+f(-u1,u1,-u1)+f(-u1,-u1,u1))...
 +w4*(f(u4,u4,v4)+f(u4,v4,u4)+f(v4,u4,u4)+f(u4,u4,-v4)+f(u4,-v4,u4)...
      +f(-v4,u4,u4)+f(-u4,u4,v4)+f(-u4,v4,u4)+f(v4,-u4,u4)+f(-u4,u4,-v4)...
      +f(-u4,-v4,u4)+f(-v4,-u4,u4)+f(u4,-u4,v4)+f(u4,v4,-u4)+f(v4,u4,-u4)...
      +f(u4,-u4,-v4)+f(u4,-v4,-u4)+f(-v4,u4,-u4)+f(-u4,-u4,v4)+f(-u4,v4,-u4)...
      +f(v4,-u4,-u4)+f(-u4,-u4,-v4)+f(-u4,-v4,-u4)+f(-v4,-u4,-u4))...
 +w5*(f(u5,v5,v5)+f(v5,u5,v5)+f(v5,v5,u5)+f(-u5,v5,v5)+f(v5,-u5,v5)...
      +f(v5,v5,-u5)+f(u5,-v5,v5)+f(-v5,u5,v5)+f(-v5,v5,u5)+f(-u5,-v5,v5)...
      +f(-v5,-u5,v5)+f(-v5,v5,-u5)+f(u5,v5,-v5)+f(v5,u5,-v5)+f(v5,-v5,u5)...
      +f(-u5,v5,-v5)+f(v5,-u5,-v5)+f(v5,-v5,-u5)+f(u5,-v5,-v5)+f(-v5,u5,-v5)...
      +f(-v5,-v5,u5)+f(-u5,-v5,-v5)+f(-v5,-u5,-v5)+f(-v5,-v5,-u5))
(I-Iexact)/Iexact   %result: -1.2762e-13
```

```
%%%%Optimal quadrature rule on CS4, and evaluation of the relative error
N=4
u1=1/sqrt(3);w1=729*pi/20020;%vertex of the cube
u2=1/sqrt(2);w2=512*pi/15015;%center of an edge
w3=736*pi/15015;%center of a face
s=sqrt(2)-1;
u4=1/sqrt(2+s^2);v4=s*u4;w4=2053*pi/51480-183*pi*sqrt(2)/80080;%edge
u5=1/sqrt(1+2*s^2);v5=s*u5;w5=2053*pi/51480+183*pi*sqrt(2)/80080;%diagonal
u6=1/sqrt(1+s^2);v6=s*u6;w6=2048*pi/45045;%equator
I=w1*(f(u1,u1,u1)+f(-u1,-u1,-u1)+f(-u1,u1,u1)+f(u1,-u1,u1)...
      +f(u1,u1,-u1)+f(u1,-u1,-u1)+f(-u1,u1,-u1)+f(-u1,-u1,u1))...
 +w2*(f(u2,u2,0)+f(u2,0,u2)+f(0,u2,u2)+f(-u2,u2,0)...
      +f(-u2,0,u2)+f(0,-u2,u2)+f(u2,-u2,0)+f(u2,0,-u2)...
      +f(0,u2,-u2)+f(-u2,-u2,0)+f(-u2,0,-u2)+f(0,-u2,-u2))+...
 +w3*(f(1,0,0)+f(0,1,0)+f(0,0,1)+f(-1,0,0)+f(0,-1,0)+f(0,0,-1))...
 +w4*(f(u4,u4,v4)+f(u4,v4,u4)+f(v4,u4,u4)+f(u4,u4,-v4)+f(u4,-v4,u4)...
      +f(-v4,u4,u4)+f(-u4,u4,v4)+f(-u4,v4,u4)+f(v4,-u4,u4)+f(-u4,u4,-v4)...
      +f(-u4,-v4,u4)+f(-v4,-u4,u4)+f(u4,-u4,v4)+f(u4,v4,-u4)+f(v4,u4,-u4)...
      +f(u4,-u4,-v4)+f(u4,-v4,-u4)+f(-v4,u4,-u4)+f(-u4,-u4,v4)+f(-u4,v4,-u4)...
      +f(v4,-u4,-u4)+f(-u4,-u4,-v4)+f(-u4,-v4,-u4)+f(-v4,-u4,-u4))...
 +w5*(f(u5,v5,v5)+f(v5,u5,v5)+f(v5,v5,u5)+f(-u5,v5,v5)+f(v5,-u5,v5)...
      +f(v5,v5,-u5)+f(u5,-v5,v5)+f(-v5,u5,v5)+f(-v5,v5,u5)+f(-u5,-v5,v5)...
      +f(-v5,-u5,v5)+f(-v5,v5,-u5)+f(u5,v5,-v5)+f(v5,u5,-v5)+f(v5,-v5,u5)...
      +f(-u5,v5,-v5)+f(v5,-u5,-v5)+f(v5,-v5,-u5)+f(u5,-v5,-v5)+f(-v5,u5,-v5)...
      +f(-v5,-v5,u5)+f(-u5,-v5,-v5)+f(-v5,-u5,-v5)+f(-v5,-v5,-u5))...
 +w6*(f(u6,v6,0)+f(u6,0,v6)+f(0,u6,v6)+f(-u6,-v6,0)+f(-u6,0,-v6)...
      +f(0,-u6,-v6)+f(-u6,v6,0)+f(-u6,0,v6)+f(0,-u6,v6)+f(u6,-v6,0)...
      +f(u6,0,-v6)+f(0,u6,-v6)+f(v6,u6,0)+f(v6,0,u6)+f(0,v6,u6)...
      +f(-v6,-u6,0)+f(-v6,0,-u6)+f(0,-v6,-u6)+f(-v6,u6,0)+f(-v6,0,u6)...
      +f(0,-v6,u6)+f(v6,-u6,0)+f(v6,0,-u6)+f(0,v6,-u6))
(I-Iexact)/Iexact   %result: 0
```

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