DISCRETE FOURTH-ORDER STURM-LIOUVILLE PROBLEMS

M. BEN-ARTZI, J.-P. CROISILLE, D. FISHELOV, AND R.KATZIR

Abstract. A discrete fourth-order elliptic theory on a one-dimensional interval is constructed. It is based on "Hermitian derivatives" and compact higher-order finite difference operators and is shown to possess the analogs of the standard elliptic theory such as coercivity and compactness. The discrete version of the fourth-order Sturm-Liouville problem $\left(\frac{d}{dx}\right)^4 u + \frac{d}{dx}\left(A(x)\frac{d}{dx}u\right) + B(x)u = f$ on a real interval is studied in terms of the functional calculus. The resulting (compact) finite difference scheme constitutes a scale of finite-dimensional Sturm-Liouville problems. A major difficulty is the presence of boundaries, in contrast to periodic problems (and analogous to boundary layers in Navier-Stokes simulations). Convergence of the finite-dimensional solutions to the continuous one is proved in the general case, and optimal ($O(h^4)$) convergence rates are obtained in the constant coefficient case. Numerical examples are given, demonstrating the optimal rate even in highly oscillatory cases.

1. INTRODUCTION

In this paper we expound a discrete elliptic theory in the context of fourth-order Sturm-Liouville problems on the interval $\Omega = [0, 1]$. The discrete finite-difference operators are compact, and are derived from the fundamental concept of the **Hermitian derivative.** It should be pointed out that the elliptic finite-difference methodology is entirely developed in the discrete framework, independently of the classical (continuous) elliptic theory. In particular, the concepts of classical elliptic theory, such as coercivity, compactness (Rellich's theorem) and apriori estimates have their equivalents in the discrete case .

One can compare the present study to the development of finite-dimensional finite element methods for elliptic problems [5].

Once the discrete structure is established, it can be applied towards the approximation of the fourth-order boundary value problem on the interval. The elliptic tools enable us to get "optimal" error estimates, as will be further explained below in this Introduction.

Our approach is closely related to recently introduced compact schemes in the treatment of $2D$ Navier-Stokes equations [3], where the pure streamfunction formulation involves fourth-order derivatives.

Date: June 29, 2017.

2010 *Mathematics Subject Classification.* Primary 65L10; Secondary 65L70, 34K28.

Key words and phrases. discrete elliptic operator, fourth order, biharmonic, Sturm-Liouville, optimal convergence, boundary values.

We thank the two anonymous referees whose very valuable observations greatly helped us in improving the presentation and clarity of the paper.

Naturally, the development of the elliptic discrete methodology involves some lengthy proofs. The reader who is primarily interested in the approximation algorithm can conveniently skip the proofs, as indicated in the "box" at the end of this Introduction.

Turning to the approximation issue, consider the equation

(1.1)
$$L_{A,B}u = \Big(\frac{d}{dx}\Big)^4 u + A(x)\Big(\frac{d}{dx}\Big)^2 u + A'(x)\Big(\frac{d}{dx}\Big)u + B(x)u = f, \quad x \in \Omega = [0,1],$$

where $A(x), B(x)$ are real functions, $A(x) \in C^1(\Omega)$ and $B(x) \in C(\Omega)$.

The equation is supplemented with homogeneous boundary conditions

$$u(0) = \frac{d}{dx}u(0) = u(1) = \frac{d}{dx}u(1) = 0. \tag{1.2}$$

As is well-known, non-homogeneous boundary conditions are accommodated by a modification of the right-hand side function $f(x)$.

The case of a second-order equation is generally known as the **Sturm-Liouville** problem. It has been extensively studied, both from the theoretical point-of-view [7], dealing with issues of spectral structure, behavior of eigenfunctions and their zeros and so on, as well as the numerical point-of-view [15], dealing with discrete aspects of these topics. We mention in particular the very recent paper [19] (and references therein), where group-theoretic tools are used for the discrete approximation of eigenvalues and eigenfunctions.

Equations such as (1.1), subject to boundary conditions at the two endpoints, are usually referred to as **higher order Sturm-Liouville problems**. Such problems appear in various applications, such as elasticity theory, streamfunction formulation of Navier-Stokes equations or wave propagation problems with high dispersivity. If restricted to the self-adjoint case these are actually one-dimensional elliptic boundary value problems, for which the basic theory is well-established. In Section 2 we recall some basic facts that are relevant to the present paper.

The "heart of the paper", the elliptic discrete analysis, is developed in Section 3. It is designed not only to the regular interior elliptic properties (such as coercivity), but also to the handling of boundary values. This additional aspect complicates the treatment, but it is certainly necessary if approximation of boundary value problems is desired. Here we encounter phenomena of "discrete boundary layer", such as lower regularity and the fact that certain operators do not commute.

In the context of elliptic boundary value problems, in any space dimension, a basic issue is the **continuous dependence** of the solutions on the data. For example, how solutions vary as the right-hand side function f is perturbed. A fundamental tool is the **compactness** of the solution operator. More specifically, it is the compact embedding (Rellich's theorem) of the Sobolev space $H^k(\Omega)$, $k \geq 1$ in $L^2(\Omega)$ [9, Chapter 5]. Roughly speaking, it is first established that the inverse of the operator (if it has no eigenvalue at zero) is bounded ("stability"). Then the compactness property is used in order to show that, under continuous variation of the data , the corresponding solutions (already shown to belong to a bounded set) vary continuously.

In a finite-dimensional space every linear operator is compact. The discrete approximation methodology is based on a sequence of finite-dimensional spaces with *increasing* dimension. A nice description of the situation is given in [24] (in the context of computing discrete approximation of spectra): "A suitable error analysis must overcome the difficulty that the solutions are in an infinite dimensional space, whereas the approximating solutions are finite dimensional vectors".

An essential feature of our calculus consists of getting operator bounds that are *independent of the mesh size*. It is in this context that we need to define the concept of **compactness in an increasing sequence of finite-dimensional spaces**. This concept is introduced in Theorem 3.7.

The discrete functional calculus leads to a finite difference scheme for the approximation of (1.1). In general terms, the scheme produces a sequence of discrete (namely, finitely valued) solutions. As the underlying mesh is refined, it is expected that the discrete solutions "converge" to the analytical one. Since these are all finite-dimensional solutions (with increasing dimension as the mesh is refined)–one needs to clarify the meaning of such convergence.

It turns out that Theorem 3.7, as in the analytical case, is the cornerstone of the convergence proof, which is expounded in Section 4; we show that, for sufficiently small mesh size h, the finite-difference scheme (4.1) can be solved (namely, the discrete operator is invertible) and indeed the resulting solutions converge to the solution of the continuous equation as $h \to 0$.

Following the general convergence proof, we consider in Section 5 quantitative error estimates for the discrete solutions, in the constant coefficient case ($A(x) \equiv a$, $B(x) \equiv b$). These are estimates of the deviation of the discrete solution from the exact one. The latter is represented by its restriction to the grid and the estimates are expressed in terms of powers of h, the mesh size. The treatment here is a crescendo process. We first establish the general Theorem 5.2; it is the exact discrete elliptic analog to the continuous case, estimating the solution and its derivatives in terms of the right-hand side. When dealing with **periodic boundary conditions,** this would have been the "end of the story", leading automatically to optimal convergence rates. However, the presence of boundary conditions (1.2) entails deterioration of the truncation error near the boundary. This in turn allows only a "suboptimal" estimate in Theorem 5.3. Remarkably, the discrete elliptic properties of the operator enable us to recover, in Theorem 5.7, an optimal $O(h^4)$ estimate (but just for the error) . In Corollary 5.9 we obtain estimates for the (discrete) derivatives of the error. As can be expected, such estimates are not quite $O(h^4)$, but they are nonetheless significant as they ensure that the discrete approximations are indeed close to the analytic solutions and do not develop spurious or oscillatory behavior.

In Section 6 we present numerical test cases that indeed corroborate our claim of optimal error estimates. This is true even for highly oscillatory solutions, such as Equation (6.9) , with variable coefficients given by (6.10).

Some of these calculations were carried out in the M.Sc. thesis of Ron Katzir, supervised by M. Ben-Artzi.

JUST THE ALGORITHM: The reader who is interested primarily in the numerical algorithm can read only Subsection 3.1
for the definitions of the discrete operators
and then Equation (4.1) for the discrete algorithm.

1.1. **Existing literature on approximations to fourth-order boundary value problems.** There is a vast literature on the numerical resolution of elliptic partial differential equations (finite-elements, finite differences, spectral methods...), and it is of course impossible for us (and beyond the scope of the paper) to give a reasonable survey. We mention the recent book [16], where Chapter 2 is devoted to elliptic problems. More specifically, Section 2.7 there deals with error analysis of fourth-order equations in the two-dimensional square, using Sobolev norms and energy methods.

Numerical studies of the biharmonic equation in a square are more relevant to our interest here, especially when they deal with issues of high-order accuracy. We refer to [1] (cubic splines collocation), [6] (finite elements) and references therein.

Finally, we focus on the one-dimensional case, that is the topic of the present paper. Generally speaking, it is fair to state that the numerical treatment for higher order Sturm-Liouville problems has attracted little attention in the literature, when compared to the classical second-order problem. The papers [13, 18, 20, 25] obtain approximate solutions to the fourth-order boundary value problem by Galerkin methods (based on B-splines). The papers [17, 23] use *quintic splines* but claim to get only second-order convergence.

We note that in all the above papers, the equation considered was $y^{(4)}(x) + g(x)y(x) = f(x)$, and in particular the second-order derivative y'' is missing.

Roughly speaking, studies of this problem are motivated, for the most part, by either one of the two following topics (that are interrelated).

- Determination of eigenvalues of the biharmonic operator and elliptic perturbations thereof [4, 6, 12, 21, 26] and references therein.
- Convergence analysis of discrete schemes for the approximation of time evolution of PDE's of mathematical physics, involving the biharmonic operator as the principal spatial part. In this category we have two-dimensional elasticity theory and the two-dimensional Navier-Stokes system in streamfunction formulation.

  Even though our convergence analysis is time-independent and confined to a one-dimensional interval, it is inspired by methods used in [2, 10, 14], invoking discrete elliptic tools such as coercivity and compact embedding (Rellich's theorem).

2. THE FOURTH-ORDER STURM-LIOUVILLE PROBLEM ON AN INTERVAL

The basic aspects of the general theory (and in fact, for elliptic operators with constant coefficients in smooth bounded domains in any dimension) are well known [8]. We briefly recall those that are relevant to the present study, where the operator $L_{A,B}$ (1.1) is defined on the closed interval $\Omega = [0, 1]$.

(1) The operator $L_{A,B}$ defined initially on $C_0^\infty(0,1)$ functions can be extended as a self-adjoint operator in $H^4(\Omega)$, the Sobolev space of functions having derivatives (in the sense of distributions) up to fourth-order in $L^2(\Omega)$.

    Its domain in this space (reflecting the homogeneous boundary conditions (1.2)) is $H^4 \cap H_0^2$, where $H_0^2(\Omega)$ is the completion of $C_0^\infty(0,1)$ in the H^2 norm.

(2) The operator $\left(\frac{d}{dx}\right)^4$ (obtained from $L_{A,B}$ when $A = B = 0$) is positive with compact resolvent $\left(\frac{d}{dx}\right)^{-4}$. Therefore its spectrum consists of an increasing

sequence of positive eigenvalues, which we designate as

$$\{0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_k ...\}.$$

(3) The lower order part $A(x)\left(\frac{d}{dx}\right)^2 + A'(x)\left(\frac{d}{dx}\right) + B(x)$ of $L_{A,B}$ is compact with respect to $\left(\frac{d}{dx}\right)^4$, hence the spectrum $\Sigma(L_{A,B})$ of $L_{A,B}$ consists also of an increasing sequence of real eigenvalues of finite multiplicity.

SPECTRAL ASSUMPTION. ***We assume that***

(2.1) $$0 \notin \Sigma(L_{A,B}).$$

(2.2) ***NOTE: in this case*** $L^{-1}_{A,B}$ ***is a compact operator on*** $L^2(\Omega)$.

3. DISCRETE FUNCTIONAL CALCULUS

3.1. **Basic setup and definition of the discrete operators.** We equip the interval $\Omega = [0, 1]$ with a uniform grid

$$x_j = jh, \quad 0 \le j \le N, \quad h = \frac{1}{N}.$$

The approximation is carried out by grid functions $\mathfrak{v}$ defined on $\{x_j,\, 0 \le j \le N\}$. The space of these grid functions is denoted by l_h^2. For their components we use either $\mathfrak{v}_j$ or $\mathfrak{v}(x_j)$.

For every smooth function $f(x)$ we define its associated grid function

(3.1) $$f_j^* = f(x_j), \quad 0 \le j \le N.$$

The discrete l_h^2 scalar product is defined by

$$(\mathfrak{v}, \mathfrak{w})_h = h \sum_{j=0}^{N} \mathfrak{v}_j \mathfrak{w}_j,$$

and the corresponding norm is

(3.2) $$|\mathfrak{v}|_h^2 = h \sum_{j=0}^{N} \mathfrak{v}_j^2.$$

For linear operators $\mathcal{A} : l_h^2 \to l_h^2$ we use $|\mathcal{A}|_h$ to denote the operator norm.

The discrete sup-norm is

(3.3) $$|\mathfrak{v}|_\infty = \max_{0 \le j \le N} \{|\mathfrak{v}_j|\}.$$

The discrete homogeneous space of grid functions is defined by

$$l_{h,0}^2 = \{\mathfrak{v},\ \mathfrak{v}_0 = \mathfrak{v}_N = 0\}.$$

Given $\mathfrak{v} \in l_{h,0}^2$ we introduce the basic (central) finite difference operators

(3.4)
$$(\delta_x \mathfrak{v})_j = \frac{1}{2h}(\mathfrak{v}_{j+1} - \mathfrak{v}_{j-1}), \quad 1 \le j \le N-1,$$
$$(\delta_x^2 \mathfrak{v})_j = \frac{1}{h^2}(\mathfrak{v}_{j+1} - 2\mathfrak{v}_j + \mathfrak{v}_{j-1}), \quad 1 \le j \le N-1,$$

The cornerstone of our approach to finite difference operators is the introduction of the **Hermitian derivative** of $\mathfrak{v} \in l_{h,0}^2$, that will replace δ_x. It will serve not only in approximating (to fourth-order of accuracy) first-order derivatives, but also as a fundamental building block in the construction of finite difference approximations to higher-order derivatives.

First, we introduce the “Simpson operator”

$$(\sigma_x \mathfrak{v})_j = \frac{1}{6}\mathfrak{v}_{j-1} + \frac{2}{3}\mathfrak{v}_j + \frac{1}{6}\mathfrak{v}_{j+1}, \quad 1 \le j \le N-1. \tag{3.5}$$

Note the operator relation (valid in $l^2_{h,0}$)

$$\sigma_x = I + \frac{h^2}{6}\delta_x^2, \tag{3.6}$$

so that σ_x is an “approximation to identity” in the following sense.

Let $\psi \in C_0^\infty(\Omega)$, then

$$|(\sigma_x - I)\psi^*|_\infty \le Ch^2\|\psi''\|_{L^\infty(\Omega)}, \tag{3.7}$$

which yields

$$|(\sigma_x - I)\psi^*|_h \le Ch^2\|\psi''\|_{L^\infty(\Omega)}. \tag{3.8}$$

In the above estimates the constant $C > 0$ is independent of h, ψ.

The Hermitian derivative $\mathfrak{v}_x$ is now defined by

$$(\sigma_x \mathfrak{v}_x)_j = (\delta_x \mathfrak{v})_j, \quad 1 \le j \le N-1. \tag{3.9}$$

Remark 3.1. *In the definition* (3.9)*, the values of* $(\mathfrak{v}_x)_j$*,* $j = 0, N$*, need to be provided, in order to make sense of the left-hand side (for* $j = 1, N-1$*). If not otherwise specified, we shall henceforth assume that, in accordance with the boundary condition* (1.2)*,* $\mathfrak{v}_x \in l^2_{h,0}$*, namely*

$$(\mathfrak{v}_x)_0 = (\mathfrak{v}_x)_N = 0.$$

In particular, the linear correspondence $l^2_{h,0} \ni \mathfrak{v} \to \mathfrak{v}_x \in l^2_{h,0}$ *is well defined, but not onto, since* δ_x *has a non-trivial kernel.*

We next introduce a fourth-order replacement to the operator δ_x^2 (see [10, Equation (15)], [3, Equation (10.50)(c)]),

$$(\widetilde{\delta}_x^2 \mathfrak{v})_j = 2(\delta_x^2 \mathfrak{v})_j - (\delta_x \mathfrak{v}_x)_j, \quad 1 \le j \le N-1. \tag{3.10}$$

The biharmonic discrete operator is given by (for $\mathfrak{v}$, $\mathfrak{v}_x \in l^2_{h,0}$),

$$\delta_x^4 \mathfrak{v} = \frac{12}{h^2}[\delta_x \mathfrak{v}_x - \delta_x^2 \mathfrak{v}]. \tag{3.11}$$

Note that, in accordance with Remark 3.1 the operator $\widetilde{\delta}_x^2$ is defined on grid functions $\mathfrak{v} \in l^2_{h,0}$, such that also $\mathfrak{v}_x \in l^2_{h,0}$.

The connection between the two difference operators for the second-order derivative is given by

$$-\widetilde{\delta_x}^2 = -\delta_x^2 + \frac{h^2}{12}\delta_x^4. \tag{3.12}$$

Remark 3.2. *Clearly the operators* δ_x*,* δ_x^2*,* δ_x^4 *depend on* h*, but for notational simplicity this dependence is not explicitly indicated.*

The fact that the biharmonic discrete operator δ_x^4 is positive (in particular symmetric) is proved in [3, Lemmas 10.9, 10.10]. Therefore its inverse $\left(\delta_x^4\right)^{-1}$ is also positive.

A fundamental tool (analogous to classical elliptic theory) is the coercivity property (with $C > 0$ independent of h) [3, Propositions 10.11,10.13],

$$(\delta_x^4 \mathfrak{z}, \mathfrak{z})_h \geq C(|\mathfrak{z}|_h^2 + |\delta_x^2 \mathfrak{z}|_h^2 + |\delta_x \mathfrak{z}_x|_h^2), \tag{3.13}$$

valid for any grid function $\mathfrak{z} \in l_{h,0}^2$ such that also $\mathfrak{z}_x \in l_{h,0}^2$.

3.2. **Uniform boundedness of the discrete operators.** We first show that , in " operator sense", the second-order operator δ_x^2 is comparable (independently of $h > 0$) to $(\delta_x^4)^{\frac{1}{2}}$.

Lemma 3.3. *The operators* $\left(\delta_x^4\right)^{-\frac{1}{2}} \delta_x^2$ *and* $\delta_x^2 \left(\delta_x^4\right)^{-\frac{1}{2}}$ *are bounded in* $l_{h,0}^2$, *with bounds that are independent of* h.

Proof. We use the coercivity property (3.13) with $\mathfrak{z} = \left(\delta_x^4\right)^{-\frac{1}{2}} \mathfrak{w}$, and obtain

$$\left(\left(\delta_x^4\right)^{\frac{1}{2}} \mathfrak{w}, \left(\delta_x^4\right)^{-\frac{1}{2}} \mathfrak{w}\right)_h \geq C \left|\delta_x^2 \left(\delta_x^4\right)^{-\frac{1}{2}} \mathfrak{w}\right|_h^2. \tag{3.14}$$

The operator $\delta_x^2 \left(\delta_x^4\right)^{-\frac{1}{2}}$ is therefore bounded, with a bound that is independent of h. The same is true (with the same bound, by a well-known fact about norms of adjoints) for its adjoint, namely, $\left(\delta_x^4\right)^{-\frac{1}{2}} \delta_x^2$. □

In the sequel we shall find it useful to use slightly different (and in fact weaker) boundedness facts (again uniform with respect to h), that are listed in the following proposition .

Proposition 3.4. *The operators* $(\delta_x^4)^{-1}$, $-\left(\delta_x^4\right)^{-1} \widetilde{\delta}_x^2$ *and* $-\widetilde{\delta}_x^2 \left(\delta_x^4\right)^{-1}$ *are bounded in* $l_{h,0}^2$, *with bounds that are independent of* h.

Proof. The boundedness of $\left(\delta_x^4\right)^{-1}$ follows directly from the coercivity property (3.13), by an obvious application of the Cauchy-Schwarz inequality.

In view of (3.12),

$$\begin{aligned} -\left(\delta_x^4\right)^{-1} \widetilde{\delta}_x^2 &= -\left(\delta_x^4\right)^{-1} \delta_x^2 + \frac{h^2}{12} \left(\delta_x^4\right)^{-1} \delta_x^4 \\ &= -\left(\delta_x^4\right)^{-1} \delta_x^2 + \frac{h^2}{12} I. \end{aligned} \tag{3.15}$$

It therefore suffices to prove the boundedness of $\left(\delta_x^4\right)^{-1} \delta_x^2$. But this simply follows from Lemma 3.3 and

$$\left(\delta_x^4\right)^{-1} \delta_x^2 = \left(\delta_x^4\right)^{-\frac{1}{2}} \left(\delta_x^4\right)^{-\frac{1}{2}} \delta_x^2.$$

□

Remark 3.5. *We can actually get explicit bounds for the operators in Proposition 3.4 as follows.*

Let $\mathfrak{z}$, $\mathfrak{z}_x \in l^2_{h,0}$. *The discrete Poincaré inequality* [3, Equation (9.37)] *yields*

$$|\mathfrak{z}|_h^2 \le |\delta_x^2 \mathfrak{z}|_h^2, \tag{3.16}$$

and from [3, Proposition 10.13] *we have*

$$|\delta_x^2 \mathfrak{z}|_h^2 \le \frac{8}{3}(\delta_x^4 \mathfrak{z}, \mathfrak{z})_h. \tag{3.17}$$

In view of the Cauchy-Schwarz inequality the second estimate implies

$$|\delta_x^2 \mathfrak{z}|_h^2 \le \frac{8}{3}|\delta_x^4 \mathfrak{z}|_h |\mathfrak{z}|_h.$$

and combined with (3.16)

$$|\mathfrak{z}|_h \le \frac{8}{3}|\delta_x^4 \mathfrak{z}|_h. \tag{3.18}$$

Also, taking $\mathfrak{z} = (\delta_x^4)^{-1}\mathfrak{w}$ *in* (3.17) *we get*

$$\Big|\delta_x^2(\delta_x^4)^{-1}\mathfrak{w}\Big|_h^2 \le \frac{8}{3}\Big(\mathfrak{w}, (\delta_x^4)^{-1}\mathfrak{w}\Big)_h \le \Big(\frac{8}{3}\Big)^2 |\mathfrak{w}|_h^2. \tag{3.19}$$

In conjunction with (3.12) *this estimate entails*

$$\Big|\widetilde{\delta}_x^2(\delta_x^4)^{-1}\mathfrak{w}\Big|_h \le \Big(\frac{8}{3} + \frac{h^2}{12}\Big)|\mathfrak{w}|_h. \tag{3.20}$$

The adjoint operator $(\delta_x^4)^{-1}\widetilde{\delta}_x^2$ *has the same bound.*

3.3. **Compactness-the discrete version of Rellich's theorem.** In (2.2) we noted the compactness of the inverse $L^{-1}_{A,B}$. The compactness of the inverse of an elliptic operator is equivalent (by domain considerations) to the compact embedding of the Sobolev space H^s, $s > 0$ in L^2. This is the celebrated Rellich theorem [9, Chapter 5.7], which is the cornerstone of the elliptic theory. Its proof requires several tools (for example, in a popular version of the proof, the use of Fourier transform and the Arzela-Ascoli theorem).

In the discrete framework we do not have some of the aforementioned analytical tools. Yet we can ask ourselves the following question.

QUESTION: Is there a suitable "compactness" property of the inverse $\Big(\delta_x^4\Big)^{-1}$?

Of course, if we just consider a *fixed* $h > 0$ such a question is meaningless since the underlying space is finite dimensional. However , we can provide a meaningful answer if *all values* of $h > 0$ are considered. In some sense, the compactness property is related to an "increasing sequence of finite-dimensional spaces". The proof is, understandably, quite long.

We first introduce some notation, basically relating grid functions to functions defined on the interval $\Omega = [0, 1]$ (see [3, Section 10.2]):

For a grid function $\mathfrak{z} \in l^2_{h,0}$ we define its associated piecewise linear continuous function by

Definition 3.6.

$$z_h(x) = \begin{cases} \text{linear in the interval } K_{i+\frac{1}{2}} = (x_i, x_{i+1}),\ 0 \le i \le N-1, \\ \mathfrak{z}_i,\ x = x_i,\ 0 \le i \le N. \end{cases}$$

Theorem 3.7. *[**The discrete Rellich theorem**] Let $\{0 < N_1 < N_2 < ...N_k < ...\}$ be an increasing sequence of integers and denote $h_k = \frac{1}{N_k}$, $k = 1, 2, ...$ Let $\left\{\mathfrak{v}^{(k)} \in l^2_{h_k,0},\ k = 1, 2, ...\right\}$ be a bounded sequence of vectors so that*

$$\sup\left\{|\mathfrak{v}^{(k)}|_{h_k},\ k = 1, 2, ...\right\} < \infty, \tag{3.21}$$

and let

$$\left\{\mathfrak{g}^{(k)} = \left(\delta_x^4\right)^{-1}(\mathfrak{v}^{(k)}),\ k = 1, 2, ...\right\}.$$

Let $\{g_{h_k}, v_{h_k}\}_{k=1}^{\infty}$ be the piecewise linear continuous functions in $\Omega = [0, 1]$ corresponding to $\{\mathfrak{g}^{(k)}, \mathfrak{v}^{(k)}\}_{k=1}^{\infty}$, respectively (Definition 3.6).

In addition, let $\left\{\mathfrak{g}_x^{(k)}\right\}_{k=1}^{\infty}$ be the sequence of Hermitian derivatives of $\{\mathfrak{g}^{(k)}\}_{k=1}^{\infty}$ and let $\{p_{h_k}\}_{k=1}^{\infty}$ be the piecewise linear continuous functions in $\Omega = [0, 1]$ corresponding to $\left\{\mathfrak{g}_x^{(k)}\right\}_{k=1}^{\infty}$. Then there exist subsequences

$$\left\{g_j := g_{h_{k_j}},\ p_j := p_{h_{k_j}},\ v_j := v_{h_{k_j}}\right\}_{j=1}^{\infty}$$

and limit functions $g(x)$, $p(x)$, $v(x)$, such that

$$\lim_{j\to\infty} g_j(x) = g(x) \text{ in } C(\Omega), \tag{3.22}$$

$$\lim_{j\to\infty} p_j(x) = p(x) \text{ in } C(\Omega), \tag{3.23}$$

$$\lim_{j\to\infty} v_j(x) = v(x) \text{ weakly in } L^2(\Omega). \tag{3.24}$$

The limit function $g(x)$ is in $H^4(\Omega) \cap H_0^2(\Omega)$ and its derivatives satisfy

$$g'(x) = p(x), \quad \left(\frac{d}{dx}\right)^4 g(x) = v(x). \tag{3.25}$$

Proof. In view of Proposition 3.4 both sequences of norms

$$\left\{|\mathfrak{g}^{(k)}|_{h_k}\right\}_{k=1}^{\infty},\ \left\{|\delta_x^2\mathfrak{g}^{(k)}|_{h_k}\right\}_{k=1}^{\infty}$$

are bounded by a constant $C > 0$.

We use the discrete Poincaré inequality [3, Equation (9.36)]

$$|\delta_x^2\mathfrak{z}|_h^2 \ge h \sum_{i=0}^{N_k-1} \left(\frac{\mathfrak{z}_{i+1} - \mathfrak{z}_i}{h}\right)^2 \tag{3.26}$$

with $\mathfrak{z} = \mathfrak{g}^{(k)}$ to conclude that , by the Cauchy-Schwarz inequality,

$$\sum_{i=0}^{N_k-1} |\mathfrak{g}_{i+1}^{(k)} - \mathfrak{g}_i^{(k)}| \le N_k^{\frac{1}{2}} \left\{ \sum_{i=0}^{N_k-1} (\mathfrak{g}_{i+1}^{(k)} - \mathfrak{g}_i^{(k)})^2 \right\}^{\frac{1}{2}} \le (h_k N_k)^{\frac{1}{2}} |\delta_x^2\mathfrak{g}^{(k)}|_{h_k} \le C. \tag{3.27}$$

Recall that (see [3, Lemma 10.4]) $\|g_{h_k}\|_{L^2(\Omega)} \le |\mathfrak{g}^{(k)}|_{h_k}$ and that the total variation of g_{h_k} satisfies

$$TV(g_{h_k}) = \sum_{i=0}^{N_k-1} |\mathfrak{g}_{i+1}^{(k)} - \mathfrak{g}_i^{(k)}|.$$

The fact that $\mathfrak{g}^{(k)} \in l^2_{h,0}$ implies that $|g_{h_k}|_{L^\infty(\Omega)} \le TV(g_{h_k}) \le C$.

For any two indices $1 \le p < m \le N_k - 1$, we get as in (3.27)

$$\sum_{i=p}^{m} |\mathfrak{g}_{i+1}^{(k)} - \mathfrak{g}_i^{(k)}| \le (m-p)^{\frac{1}{2}} \Big\{ \sum_{i=0}^{N_k-1} (\mathfrak{g}_{i+1}^{(k)} - \mathfrak{g}_i^{(k)})^2 \Big\}^{\frac{1}{2}} \le [h_k(m-p)]^{\frac{1}{2}} |\delta_x^2 \mathfrak{g}^{(k)}|_{h_k} \le C|x_m - x_p|^{\frac{1}{2}}. \tag{3.28}$$

Thus, for any $0 \le x < y \le 1$,

$$|g_{h_k}(y) - g_{h_k}(x)| \le C|y-x|^{\frac{1}{2}}, \tag{3.29}$$

where $C > 0$ is independent of k.

It follows that the sequence $\{g_{h_k}\}_{k=1}^\infty$ is uniformly bounded and equicontinuous. The Arzela-Ascoli theorem implies that there exists a subsequence $\Big\{g_j := g_{h_{k_j}}\Big\}_{j=1}^\infty$ that converges uniformly, as asserted in (3.22).

Let $g^{(4)}$ be the fourth derivative (in the sense of distributions) of the function g and let $\phi \in C_0^\infty(0,1)$ be a test function. Denoting by $< \cdot, \cdot >$ the pairing of distributions and test functions we have

$$< g^{(4)}, \phi >= \int_0^1 g(x)\phi^{(4)}(x)dx = \lim_{j\to\infty} \int_0^1 g_j(x)\phi^{(4)}(x)dx. \tag{3.30}$$

Let $\phi_j^{(4)}(x) = \phi_{h_{k_j}}^{(4)}(x)$ be the piecewise linear continuous function corresponding to $(\phi^{(4)})^*$ (on the grid with mesh size h_{k_j}). Clearly the sequence $\Big\{\phi_j^{(4)}\Big\}_{j=1}^\infty$ converges uniformly to $\phi^{(4)}$, so that

$$\lim_{j\to\infty} \int_0^1 g_j(x)\phi^{(4)}(x)dx = \lim_{j\to\infty} \int_0^1 g_j(x)\phi_j^{(4)}(x)dx. \tag{3.31}$$

The integral in the right-hand side, involving only piecewise linear functions, can be expressed as (see [3, Lemma 10.4])

$$\int_0^1 g_j(x)\phi_j^{(4)}(x)dx = (\mathfrak{g}^{(k_j)}, (\phi_j^{(4)})^*)_{h_{k_j}} - \frac{h_{k_j}}{6} \sum_{m=0}^{N_{k_j}-1} (\mathfrak{g}_{m+1}^{(k_j)} - \mathfrak{g}_m^{(k_j)})((\phi_j^{(4)})^*_{m+1} - (\phi_j^{(4)})^*_m) \tag{3.32}$$

where $(\phi_j^{(4)})^* \in l^2_{h_{k_j},0}$ is the grid function associated with the function $\phi^{(4)}(x)$ (with mesh size h_{k_j}).

Clearly

$$\max_{0\le m\le N_{k_j}-1} |(\phi_j^{(4)})^*_{m+1} - (\phi_j^{(4)})^*_m| \xrightarrow[j\to\infty]{} 0,$$

so that in view of (3.27) we obtain from (3.32)

$$\lim_{j\to\infty} \int_0^1 g_j(x)\phi_j^{(4)}(x)dx = \lim_{j\to\infty} (\mathfrak{g}^{(k_j)}, (\phi_j^{(4)})^*)_{h_{k_j}}. \tag{3.33}$$

We now invoke the estimate (see [3, Proposition 10.8])

$$|\delta_x^4\phi^* - (\phi_j^{(4)})^*|_{h_{k_j}} \le Ch_{k_j}^{\frac{3}{2}},$$

with a constant $C > 0$ depending only on ϕ. Note that δ_x^4 acts in the space $l^2_{h_{k_j},0}$.

It follows that

$$\lim_{j\to\infty} (\mathfrak{g}^{(k_j)}, (\phi_j^{(4)})^*)_{h_{k_j}} = \lim_{j\to\infty} (\mathfrak{g}^{(k_j)}, \delta_x^4\phi^*)_{h_{k_j}}, \tag{3.34}$$

and combining Equations (3.31)-(3.34) we obtain

$$\lim_{j\to\infty} \int_0^1 g_j(x)\phi^{(4)}(x)dx = \lim_{j\to\infty} (\mathfrak{g}^{(k_j)}, \delta_x^4\phi^*)_{h_{k_j}}. \tag{3.35}$$

In view of Equation (3.30) and the symmetry of δ_x^4 the last equation yields

$$< g^{(4)}, \phi >= \lim_{j\to\infty} (\delta_x^4\mathfrak{g}^{(k_j)}, \phi^*)_{h_{k_j}} = \lim_{j\to\infty} (\mathfrak{v}^{(k_j)}, \phi^*)_{h_{k_j}}. \tag{3.36}$$

We now turn to the sequence $\left\{\mathfrak{v}^{(k)} \in l^2_{h_k,0},\ k = 1, 2, ...\right\}$ and its associated sequence of piecewise linear continuous functions v_{h_k}.

Since $\|v_{h_k}\|_{L^2(\Omega)} \le |\mathfrak{v}^{(k)}|_{h_k}$, a subsequence of $\left\{v_j = v_{h_{k_j}}\right\}_{j=1}^{\infty}$ converges weakly to a function $v \in L^2(\Omega)$. We retain the notation $\{k_j\}$ for this subsequence.

Denote by $\phi_j(x) = \phi_{h_{k_j}}(x)$ the piecewise linear continuous function corresponding to ϕ^* (with mesh size h_{k_j}). As in (3.32) we have

$$(\mathfrak{v}^{(k_j)}, \phi^*)_{h_{k_j}} = \int_0^1 v_j(x)\phi_j(x)dx + \frac{h_{k_j}}{6} \sum_{m=0}^{N_{k_j}-1} (\mathfrak{v}_{m+1}^{(k_j)} - \mathfrak{v}_m^{(k_j)})((\phi_j)^*_{m+1} - (\phi_j)^*_m). \tag{3.37}$$

By the Cauchy-Schwarz inequality

$$\Big| \sum_{m=0}^{N_{k_j}-1} (\mathfrak{v}_{m+1}^{(k_j)} - \mathfrak{v}_m^{(k_j)})\Big| \le N_{k_j}^{\frac{1}{2}} \Big\{ \sum_{m=0}^{N_{k_j}-1} (\mathfrak{v}_{m+1}^{(k_j)} - \mathfrak{v}_m^{(k_j)})^2 \Big\}^{\frac{1}{2}} \le Ch_{k_j}^{-1}.$$

Also, with a constant $C > 0$ depending only on ϕ,

$$|(\phi_j)^*_{m+1} - (\phi_j)^*_m| \le Ch_{k_j},\ m = 1, 2, \dots N_{k_j},$$

so the last equation yields ,

$$\lim_{j\to\infty} (\mathfrak{v}^{(k_j)}, \phi^*)_{h_{k_j}} = \lim_{j\to\infty} \int_0^1 v_j(x)\phi_j(x)dx. \tag{3.38}$$

Since $v_j(x)$ converges weakly to $v(x)$ and $\phi_j(x)$ converges uniformly to $\phi(x)$, we get finally from Equation (3.36)

$$< g^{(4)}, \phi >= \int_0^1 v(x)\phi(x)dx. \tag{3.39}$$

By standard elliptic theory we conclude that $g \in H^4(\Omega)$, the Sobolev space of order four, and $g^{(4)} = v$. The Sobolev embedding theorem now yields

$$g \in C^3(0, 1). \tag{3.40}$$

Our next goal is to obtain the boundary values of $g(x)$. This will be carried out by establishing the limit (3.23) (taking a further subsequence if needed).

By the definition of the inverse operator $\left(\delta_x^4\right)^{-1}$ we know that

$$\mathfrak{g}_0^{(k)} = \mathfrak{g}_{N_k}^{(k)} = (\mathfrak{g}_x^{(k)})_0 = (\mathfrak{g}_x^{(k)})_{N_k} = 0, \quad k = 1, 2, \dots, \tag{3.41}$$

and we need to establish similar values for g.

From the uniform convergence (3.22) and the fact that $g_j(0) = g_j(1) = 0$, $j = 1, 2, \dots$ we obtain

$$g(0) = g(1) = 0. \tag{3.42}$$

In order to deal with the boundary values of $g'(x)$ we consider the sequence of grid functions $\left\{\mathfrak{g}_x^{(k)}\right\}_{k=1}^{\infty}$, the Hermitian derivatives of the sequence $\left\{\mathfrak{g}^{(k)}\right\}_{k=1}^{\infty}$. Let $\left\{p_h^{(k)}(x)\right\}_{k=1}^{\infty}$ be the corresponding sequence of continuous piecewise linear functions (Definition 3.6).

In addition to (3.13) we have also the coercivity property [3, Propositions 10.11],

$$(\mathfrak{v}^{(k)}, \mathfrak{g}^{(k)})_h = (\delta_x^4 \mathfrak{g}^{(k)}, \mathfrak{g}^{(k)})_h \geq h \sum_{i=0}^{N_k-1} \Big(\frac{(\mathfrak{g}_x^{(k)})_{i+1} - (\mathfrak{g}_x^{(k)})_i}{h}\Big)^2 \tag{3.43}$$

(Compare (3.26)).

As in (3.28) we have, for any two indices $1 \leq p < m \leq N_k - 1$,

$$\begin{aligned} \sum_{i=p}^{m} |(\mathfrak{g}_x^{(k)})_{i+1} - (\mathfrak{g}_x^{(k)})_i| &\leq (m-p)^{\frac{1}{2}} \Big\{ \sum_{i=0}^{N_k-1} [(\mathfrak{g}_x^{(k)})_{i+1} - (\mathfrak{g}_x^{(k)})_i]^2 \Big\}^{\frac{1}{2}} \\ &\leq [h_k(m-p)]^{\frac{1}{2}} |\delta_x^4 \mathfrak{g}^{(k)}|_{h_k}^{\frac{1}{2}} |\mathfrak{g}^{(k)}|_{h_k}^{\frac{1}{2}} \leq C|x_m - x_p|^{\frac{1}{2}}. \end{aligned} \tag{3.44}$$

Thus, for any $0 \leq x < y \leq 1$,

$$|p_{h_k}(y) - p_{h_k}(x)| \leq C|y - x|^{\frac{1}{2}}, \tag{3.45}$$

where $C > 0$ is independent of k.

It follows that the sequence $\{p_{h_k}\}_{k=1}^{\infty}$ is uniformly bounded and equicontinuous, so the Arzela-Ascoli theorem yields the existence of a subsequence (we retain the notation k_j used above) $\left\{p_j := p_{h_{k_j}}\right\}_{j=1}^{\infty}$ that converges uniformly to a continuous function $p(x)$. Remark that $p(0) = p(1) = 0$, since this is true for all p_j.

We shall now establish the fact that

$$p(x) \equiv g'(x). \tag{3.46}$$

Let $\phi \in C_0^\infty(0,1)$ be a test function as above, and let $\phi_j(x) = \phi_{h_{k_j}}(x)$ (resp. $\phi'_j(x) = \phi'_{h_{k_j}}(x)$) be the piecewise linear continuous function corresponding to ϕ^* (resp. $(\phi')^*$). Clearly the sequence $\left\{\phi'_j\right\}_{j=1}^{\infty}$ converges uniformly to ϕ'.

As in (3.33) we get

$$\begin{aligned} \int_0^1 p(x)\phi(x)dx &= \lim_{j\to\infty} \int_0^1 p_j(x)\phi(x)dx \\ = \lim_{j\to\infty} \int_0^1 p_j(x)\phi_j(x)dx &= \lim_{j\to\infty} (\mathfrak{g}_x^{(k_j)}, \phi^*)_{h_{k_j}}. \end{aligned} \tag{3.47}$$

Invoking the definition (3.9) of the Hermitian derivative,

$$(\mathfrak{g}_x^{(k_j)}, \phi^*)_{h_{k_j}} = (\sigma_x^{-1}\delta_x \mathfrak{g}^{(k_j)}, \phi^*)_{h_{k_j}} = (\delta_x \mathfrak{g}^{(k_j)}, \sigma_x^{-1}\phi^*)_{h_{k_j}}. \tag{3.48}$$

From (3.26) we infer that

$$\sup_{j=1,2,\ldots} \left\{ |\delta_x \mathfrak{g}^{(k_j)}|_h \right\} \le C \sup_{j=1,2,\ldots} \left\{ |\delta_x^2 \mathfrak{g}^{(k_j)}|_h \right\} < \infty. \tag{3.49}$$

Also $\sigma_x^{-1}\phi^*_{h_{k_j}} - \phi^*_{h_{k_j}} = \sigma_x^{-1}[\phi^*_{h_{k_j}} - \sigma_x \phi^*_{h_{k_j}}]$, and it is known [3, Equation (10.87)] that the operator-bound of σ_x^{-1} is independent of h. Thus, noting (3.8) we infer from (3.48)

$$\begin{aligned} \lim_{j\to\infty} (\mathfrak{g}_x^{(k_j)}, \phi^*)_{h_{k_j}} &= \lim_{j\to\infty} (\delta_x \mathfrak{g}^{(k_j)}, \phi^*)_{h_{k_j}} \\ = -\lim_{j\to\infty} (\mathfrak{g}^{(k_j)}, \delta_x \phi^*)_{h_{k_j}} &= -\lim_{j\to\infty} \int_0^1 g_j(x)\phi'(x)dx = -\int_0^1 g(x)\phi'(x)dx. \end{aligned} \tag{3.50}$$

With the same arguments as those leading to Equation (3.33)) we get

$$\begin{aligned} \lim_{j\to\infty} \int_0^1 p_j(x)\phi(x)dx &= \lim_{j\to\infty} (\mathfrak{g}_x^{(k_j)}, \phi^*)_{h_{k_j}} \\ &= -\lim_{j\to\infty} \int_0^1 g_j(x)\phi'(x)dx = -\int_0^1 g(x)\phi'(x)dx. \end{aligned} \tag{3.51}$$

Combining this result with (3.47) we conclude that $g'(x) = p(x)$ and in particular $g'(0) = g'(1) = 0$.

□

In the proof of Theorem 3.7 we have seen that in addition to the convergence (3.22), the piecewise-linear functions corresponding to the Hermitian derivatives $\left\{ \mathfrak{g}_x^{(k_j)} \right\}_{j=1}^{\infty}$ converge uniformly to $g'(x)$ (3.23). Next we show that a weaker convergence statement holds for the second-order derivatives.

Corollary 3.8. *In the setting of Theorem 3.7 let*

$$\mathfrak{w}^{(k)} = \widetilde{\delta}_x^2 \mathfrak{g}^{(k)} = \widetilde{\delta}_x^2 \left(\delta_x^4\right)^{-1} (\mathfrak{v}^{(k)}).$$

Let w_{h_k} be the piecewise linear continuous functions in $\Omega = [0,1]$ corresponding to $\mathfrak{w}^{(k)}$ (Definition 3.6).

Let the sequences $\left\{ g_j := g_{h_{k_j}},\ v_j := v_{h_{k_j}} \right\}_{j=1}^{\infty}$ and limit functions $g(x),\ v(x)$, be as in theorem 3.7 and let $\left\{ w_j := w_{h_{k_j}} \right\}_{j=1}^{\infty}$.

Then

$$\lim_{j\to\infty} w_j(x) = g''(x) \ \text{weakly in}\ L^2(\Omega), \tag{3.52}$$

Proof. Let $\phi(x)$ be a test function as in the proof of Theorem 3.7. Then, with the notation used in that proof and using the definition (3.9) of the Hermitian derivative,

$$(\mathfrak{w}^{(k_j)}, \phi^*)_{h_{k_j}} = (\widetilde{\delta}_x^2 \mathfrak{g}^{(k_j)}, \phi^*)_{h_{k_j}} = (\mathfrak{g}^{(k_j)}, \widetilde{\delta}_x^2 \phi^*)_{h_{k_j}}.$$

With the same arguments as in the proof of the theorem (see in particular Equation (3.33)) we get

$$\begin{aligned}\lim_{j\to\infty}\int_0^1 w_j(x)\phi(x)dx &= \lim_{j\to\infty}(\mathfrak{w}^{(k_j)},\phi^*)_{h_{k_j}} \\ = \lim_{j\to\infty}\int_0^1 g_j(x)\phi''(x)dx &= \int_0^1 g(x)\phi''(x)dx,\end{aligned} \tag{3.53}$$

which proves (3.52). □

3.4. **Connection to the continuous case.** The fact that the boundedness assumption (3.21) deals with a *general* sequence of grid functions allowed us to get only the weak convergence result of Corollary 3.8. However, if we deal with a sequence of grid functions associated with the *same* test function, we can obtain a better result.

We first connect the discrete biharmonic operator to the continuous one by the following claim [3, Theorem 10.19] . In fact, we are using the (stronger) sup-norm estimate that is included in the proof of that theorem [3, Equation (10.167)].

Claim 3.9. *Let $f(x)$ be a smooth function in $\Omega = [0,1]$. Let $u(x)$ satisfy*

$$\left(\frac{d}{dx}\right)^4 u(x) = f(x),$$

subject to homogeneous boundary conditions (1.2)*. Then*

$$|u^* - (\delta_x^4)^{-1} f^*|_\infty = O(h^4). \tag{3.54}$$

Remark 3.10. *The "$O(h^4)$" here means that there exists a constant $C > 0$, depending only on f, such that for all integers $N > 1$,*

$$|u^* - (\delta_x^4)^{-1} f^*|_\infty \le Ch^4, \quad h = \frac{1}{N}.$$

Observe that the grid functions in this estimate are defined on the grid of (the variable) mesh size h.

We can now introduce the following improvement to the weak convergence result of Corollary 3.8.

Proposition 3.11. *Let $\phi \in C_0^\infty(0,1)$. Let $\{0 < N_1 < N_2 < ...N_k < ...\}$ be an increasing sequence of integers and denote $h_k = \frac{1}{N_k}$, $k = 1,2,...$ Let*

$$\left\{\mathfrak{v}^{(k)} = \phi_k^* \in l^2_{h_k,0},\, k = 1,2,...\right\}$$

be the bounded sequence of grid functions corresponding to $\phi(x)$ (on the sequence of grids with mesh sizes h_k).

Then, in the setting (and notations) of Corollary 3.8, we have instead of (3.52)

$$\lim_{j\to\infty} w_j(x) = g''(x) \quad \text{in } C(\Omega), \tag{3.55}$$

where $g \in H^4(\Omega)\cap H_0^2(\Omega)$ satisfies $\left(\frac{d}{dx}\right)^4 g(x) = \phi(x)$. In fact, as is seen from the proof , the whole sequence $\{w_{h_k}\}_{k=1}^\infty$ converges in the sense of (3.55)*.*

Proof. Let $g(x) \in H^4(\Omega) \cap H_0^2(\Omega)$ (and in fact it is a C^∞ function) satisfy

$$g^{(4)}(x) = \phi(x).$$

(Note that the function ϕ here is clearly the limit function v in (3.25)).

The basic optimal convergence fact in Claim 3.9 yields here

$$|g_k^* - (\delta_x^4)^{-1}\phi_k^*|_\infty \le Ch_k^4, \quad k = 1, 2, \dots, \tag{3.56}$$

where $g_k^* \in l^2_{h_k,0}$ is the grid function corresponding to g and $C > 0$ is independent of k. Observe that g_k^* is the grid function corresponding to the *continuous solution*, and thus not equal to the *approximate* grid function $\mathfrak{g}^{(k)} = (\delta_x^4)^{-1}\phi_k^*$ of Theorem 3.7.

By the definition (3.4) of δ_x^2 we get

$$|\delta_x^2 g_k^* - \delta_x^2(\delta_x^4)^{-1}\phi_k^*|_\infty \le Ch_k^2, \quad k = 1, 2, \dots,$$

and in view of (3.12) also

$$|\delta_x^2 g_k^* - \widetilde{\delta^2}_x(\delta_x^4)^{-1}\phi_k^*|_\infty = |\delta_x^2 g_k^* - \mathfrak{w}^{(k)}|_\infty \le Ch_k^2, \quad k = 1, 2, \dots,$$

where $\mathfrak{w}^{(k)}$ is as introduced in Corollary 3.8.

Replacing $\delta_x^2 g_k^*$ by $(g'')_k^* \in l^2_{h_k,0}$, the grid function corresponding to g'', the Taylor expansion yields

$$|(g'')_k^* - \mathfrak{w}^{(k)}|_\infty \le Ch_k^2, \quad k = 1, 2, \dots. \tag{3.57}$$

Let $\{(g'')_{h_k}(x)\}_{k=1}^\infty$ be the sequence of piecewise linear continuous functions in $\Omega = [0, 1]$ corresponding to $\{(g'')_k^*\}_{k=1}^\infty$ (Definition 3.6). The inequality (3.57) yields a similar one for the corresponding piecewise linear functions

$$\max_{0 \le x \le 1} |(g'')_{h_k}(x) - w_{h_k}(x)| \le Ch_k^2, \quad k = 1, 2, \dots. \tag{3.58}$$

Clearly

$$\max_{0 \le x \le 1} |g''(x) - (g'')_{h_k}(x)| \le Ch_k^2, \quad k = 1, 2, \dots,$$

and inserting this in (3.58) we infer

$$\max_{0 \le x \le 1} |g''(x) - w_{h_k}(x)| \le Ch_k^2, \quad k = 1, 2, \dots, \tag{3.59}$$

which concludes the proof of the proposition. □

4. A DISCRETE VERSION OF THE FOURTH-ORDER STURM-LIOUVILLE EQUATION

Using the finite difference operators introduced in Section 3, and taking $h = \frac{1}{N}$, we introduce the discrete analog of Equation (1.1) by

(4.1)

$$[L_{A,B,h}\mathfrak{g}^h]_i = (\delta_x^4 \mathfrak{g}^h)_i + A_i^{*,h}(\widetilde{\delta_x^2}\mathfrak{g}^h)_i + (A')_i^{*,h}(\mathfrak{g}_x^h)_i + B_i^{*,h}\mathfrak{g}_i^h = f_i^{*,h}, \quad 1 \le i \le N-1,$$

where $f^{*,h}$, $A^{*,h}$, $(A')^{*,h}$, $B^{*,h}$ are the grid functions corresponding, respectively, to $f(x)$, $A(x)$, $A'(x)$, $B(x)$.

We assume that $f(x)$ is continuous in $\Omega = [0, 1]$.

The equation is supplemented with homogeneous boundary conditions

$$\mathfrak{g}_0^h = (\mathfrak{g}_x^h)_0 = \mathfrak{g}_N^h = (\mathfrak{g}_x^h)_N = 0.$$

Thus, we seek solution $\mathfrak{g}^h \in l^2_{h,0}$, such that also $\mathfrak{g}_x^h \in l^2_{h,0}$.

Remark 4.1. *As in Remark 3.1 we assume that all grid functions and their Hermitian derivatives are in* $l^2_{h,0}$. *This amounts simply to extending the grid functions (whose relevant values are at the interior points* $\{x_i,\ 1 \le i \le N-1\}$*) as zero at the endpoints* x_0, x_N.

In what follows we designate,

$$\begin{cases} \mathfrak{g}^h_x, \quad \text{the Hermitian derivative of } \mathfrak{g}^h\,, \\ \mathfrak{v}^h = \delta_x^4 \mathfrak{g}^h, \\ \mathfrak{w}^h = \widetilde{\delta}_x^2 \mathfrak{g}^h = \widetilde{\delta}_x^2 \Big(\delta_x^4\Big)^{-1} \mathfrak{v}^h. \end{cases} \tag{4.2}$$

The basic result here is that "stability" implies "convergence" as follows.

Theorem 4.2. *[**General convergence**] Let* $\{0 < N_1 < N_2 < ...N_k < ...\}$ *be an increasing sequence of integers and denote* $h_k = \frac{1}{N_k}$, $k = 1, 2, ...$

Let $\Big\{\mathfrak{g}^{(k)} = \mathfrak{g}^{h_k} \in l^2_{h_k,0},\ k = 1, 2, ...\Big\}$ *be a sequence of solutions to Equation* (4.1) *(with* $h = h_k$*). Let* $\mathfrak{v}^{(k)} = \mathfrak{v}^{h_k}$ *and assume that* $\mathfrak{v}^{(k)}_x \in l^2_{h_k,0}$, $k = 1, 2, ...$

Assume that

$$\sup\Big\{|\mathfrak{v}^{(k)}|_{h_k} = |\delta_x^4 \mathfrak{g}^{(k)}|_{h_k},\ k = 1, 2, ...\Big\} < \infty. \tag{4.3}$$

Let g_{h_k}, v_{h_k} *be the piecewise linear continuous functions in* $\Omega = [0, 1]$ *corresponding to* $\mathfrak{g}^{(k)}$, $\mathfrak{v}^{(k)}$ *(Definition 3.6).*

Then these sequences converge to limit functions $g(x)$, $v(x)$, *in the following sense*

$$\lim_{k\to\infty} g_{h_k}(x) = g(x) \ \textit{in}\ C(\Omega), \tag{4.4}$$

$$\lim_{k\to\infty} v_{h_k}(x) = v(x) \ \textit{weakly in}\ L^2(\Omega). \tag{4.5}$$

The limit function $g(x)$ *is in* $H^4(\Omega) \cap H^2_0(\Omega)$ *and satisfies Equation* (1.1)*:*

$$L_{A,B}\, g = \Big(\frac{d}{dx}\Big)^4 g + A(x)\Big(\frac{d}{dx}\Big)^2 g + A'(x)\Big(\frac{d}{dx}\Big) g + B(x) g = f.$$

Proof. Writing Equation (4.1) in terms of the function $\mathfrak{v}^{(k)}$ yields

$$\begin{aligned} \mathfrak{v}^{(k)}_i + A^{*,h_k}_i[\widetilde{\delta}_x^2(\delta_x^4)^{-1}\mathfrak{v}^{(k)}]_i + (A')^{*,h_k}_i[((\delta_x^4)^{-1}\mathfrak{v}^{(k)})_x]_i \\ + B^{*,h_k}_i[(\delta_x^4)^{-1}\mathfrak{v}^{(k)}]_i = f^{*,h_k}_i, \quad 1 \le i \le N-1. \end{aligned} \tag{4.6}$$

The boundedness assumption (4.3) enables us to invoke Theorem 3.7 and Corollary 3.8. Thus there exist subsequences $\Big\{g_j := g_{h_{k_j}},\ v_j := v_{h_{k_j}}\Big\}_{j=1}^{\infty}$ and limit functions $g(x),\ v(x)$, such that

$$\begin{cases} \lim\limits_{j\to\infty} g_j(x) = g(x) \text{ in } C(\Omega), \\ \lim\limits_{j\to\infty} v_j(x) = v(x) \text{ weakly in } L^2(\Omega). \end{cases} \tag{4.7}$$

The limit function $g(x)$ is in $H^4(\Omega) \cap H^2_0(\Omega)$ and $\Big(\frac{d}{dx}\Big)^4 g = v$.

Denote by p_{h_k}, w_{h_k} the piecewise linear continuous functions in $\Omega = [0, 1]$ corresponding , respectively, to $\mathfrak{g}^{(k)}_x$, $\mathfrak{w}^{(k)}$ (Definition 3.6). Let $\Big\{p_j = p_{h_{k_j}},\ w_j := w_{h_{k_j}}\Big\}_{j=1}^{\infty}$.

From (3.23) and (3.52) we obtain,

$$\begin{cases} \lim_{j\to\infty} p_j(x) = g'(x) \ \text{ in } \ C(\Omega), \\ \lim_{j\to\infty} w_j(x) = g''(x) \text{ weakly in } \ L^2(\Omega). \end{cases} \tag{4.8}$$

Inserting these limits in (4.6) we conclude that the following equation is satisfied in the weak sense.

$$\Big(\frac{d}{dx}\Big)^4 g + A(x)\Big(\frac{d}{dx}\Big)^2 g + A'(x)\Big(\frac{d}{dx}\Big) g + B(x)g = f. \tag{4.9}$$

However, in view of the Assumption (2.1) there is a unique solution to this equation , so all subsequences of $\{g_{h_k},\, v_{h_k}\}_{k=1}^{\infty}$ converge to the same limit. This concludes the proof of the theorem. □

In the proof of Theorem 5.2 we shall need a variant of Theorem 4.2, keeping all the assumptions of the latter but allowing the right-hand side in Equation (4.1) to be a general decaying sequence of vectors. In the following corollary we use the notation introduced in Theorem 4.2.

Corollary 4.3. *Suppose that we have a sequence of grid functions* $\big\{\mathfrak{g}^{h_k}\big\}_{k=1}^{\infty}$, *with* $h_k \downarrow 0$ *as* $k \to \infty$, *satisfying the equation*

(4.10)

$$[L_{A,B,h}\mathfrak{g}^{h_k}]_i = (\delta_x^4 \mathfrak{g}^{h_k})_i + A_i^{*,h_k}(\widetilde{\delta}_x^2 \mathfrak{g}^{h_k})_i + (A')_i^{*,h_k}(\mathfrak{g}_x^{h_k})_i + B_i^{*,h_k}\mathfrak{g}_i^{h_k} = \mathfrak{r}_i^{h_k}, \quad 1 \le i \le N-1,$$

where

$$\lim_{k\to\infty} |\mathfrak{r}^{h_k}|_{h_k} = 0. \tag{4.11}$$

Assume that (4.3) *holds. Then*

$$\lim_{k\to\infty} g_{h_k}(x) = g(x) \ \text{ in } \ C(\Omega), \tag{4.12}$$

where the limit function $g(x)$ *is in* $H^4(\Omega) \cap H_0^2(\Omega)$ *and satisfies the equation*

$$\Big(\frac{d}{dx}\Big)^4 g + A(x)\Big(\frac{d}{dx}\Big)^2 g + A'(x)\Big(\frac{d}{dx}\Big) g + B(x)g = 0.$$

Proof. The proof follows verbatim the proof of Theorem 4.2 and in getting Equation (4.9) for the limit, the right-hand side is zero due to the assumption (4.11). □

5. ERROR ESTIMATES OF THE DISCRETE APPROXIMATION

In Theorem 4.2 we have established the convergence of the discrete solutions of (4.1) (extended as piecewise linear continuous functions) to the solutions of the differential equation (1.1).

The purpose of this section is to provide a more quantitative rate of convergence.

Remark 5.1. *It is fundamentally important to note that our estimates become complicated due to the presence of* ***boundary conditions****. If instead of the (homogeneous) boundary conditions we impose* ***periodicity conditions*** *(namely, the equation is solved on a circle) then the whole issue of estimating the error is reduced to the determination of the truncation error, which in our scheme is "optimal" (of fourth-order) as will be discussed in detail below.*

We shall carry the study under the simplifying assumption that the coefficients in (1.1) are constant, namely, there are constants $a,\, b \in \mathbf{R}$ so that

$$A(x) \equiv a, \;\; B(x) \equiv b, \quad x \in \Omega = [0,1].$$

Equation (1.1) now takes the simplified form

$$\left(\frac{d}{dx}\right)^4 u + a\left(\frac{d}{dx}\right)^2 u + bu = f, \quad x \in \Omega = [0,1]. \tag{5.1}$$

In this case Equation (4.1) takes the simpler form

$$(\delta_x^4 \mathfrak{g}^h)_i + a(\widetilde{\delta}_x^2 \mathfrak{g}^h)_i + b\mathfrak{g}_i^h = f_i^{*,h}, \quad 1 \le i \le N-1. \tag{5.2}$$

The equation is supplemented with homogeneous boundary conditions

$$\mathfrak{g}_0^h = (\mathfrak{g}_x^h)_0 = \mathfrak{g}_N^h = (\mathfrak{g}_x^h)_N = 0.$$

Thus, we seek solution $\mathfrak{g}^h \in l^2_{h,0}$, such that also $\mathfrak{g}_x^h \in l^2_{h,0}$.

Observe that

$$\sup_{N=1,2,\dots} |f^{*,h}|_h < \infty, \quad h = \frac{1}{N}.$$

5.1. **Elliptic estimates–up to the boundary.** We shall first look at the general discrete elliptic equation ,

$$\delta_x^4 \mathfrak{w}^h + a\widetilde{\delta}_x^2 \mathfrak{w}^h + b\mathfrak{w}^h = \mathfrak{r}^h, \quad \mathfrak{r}^h \in l^2_{h,0}, \tag{5.3}$$

subject to the homogeneous boundary conditions.

Note that Equation (5.2) is a special case, with the right-hand side equal to f^*.

The following theorem states that all three terms in the left-hand side of (5.3) are uniformly bounded for sufficiently small h, and in particular guarantees the bounded invertibility of the operator

$$\mathcal{L}_h = \delta_x^4 + a\widetilde{\delta}_x^2 + bI, \tag{5.4}$$

acting on grid functions $\mathfrak{v} \in l^2_{h,0}$ such that also $\mathfrak{v}_x \in l^2_{h,0}$.

The theorem to be proved is the precise analog of the global regularity estimates for elliptic operators in the continuous case [11, Section I.17].

Theorem 5.2. *[**Fundamental discrete Sobolev estimates**] Let $\mathfrak{w}^h$ be the solution to* (5.3)*. Then there exists an integer $N_0 > 1$ and a constant $C > 0$ (depending only on N_0) such that*

$$|\delta_x^4 \mathfrak{w}^h|_h + |\widetilde{\delta}_x^2 \mathfrak{w}^h|_h + |\mathfrak{w}^h|_h \le C|\mathfrak{r}^h|_h, \quad N_0 < N, \; h = \frac{1}{N}. \tag{5.5}$$

Proof. We first show the estimate for the fourth-order discrete derivative $\delta_x^4 \mathfrak{w}^h$. The estimates for the lower order terms will easily follow from that.

Suppose to the contrary that there exist sequences $h_k \to 0$ and $\left\{ \mathfrak{r}^{h_k} \in l^2_{h_k,0} \right\}_{k=1}^{\infty}$ such that $|\mathfrak{r}^{h_k}|_{h_k} = 1$ while

$$\lim_{k\to\infty} |\delta_x^4 \mathfrak{w}^{h_k}|_{h_k} = +\infty. \tag{5.6}$$

From Equation (5.3) it follows that

$$|\delta_x^4 \mathfrak{w}^{h_k}|_{h_k}^2 \leq \Big[|a||\widetilde{\delta}_x^2 \mathfrak{w}^{h_k}|_{h_k} + |b||\mathfrak{w}^{h_k}|_{h_k} + |\mathfrak{r}^{h_k}|_{h_k}\Big]^2. \tag{5.7}$$

The coercivity property (3.13) (note also (3.10)) implies that

$$|\widetilde{\delta}_x^2 \mathfrak{w}^{h_k}|_{h_k}^2 \leq \frac{1}{2(1+|a|)}|\delta_x^4 \mathfrak{w}^{h_k}|_{h_k}^2 + \frac{1+|a|}{2}|\mathfrak{w}^{h_k}|_{h_k}^2.$$

Plugging this estimate into (5.7) and recalling that $|\mathfrak{r}^{h_k}|_{h_k} = 1$ we get

$$|\delta_x^4 \mathfrak{w}^{h_k}|_{h_k}^2 \leq C[1 + |\mathfrak{w}^{h_k}|_{h_k}^2], \quad k = 1, 2, \dots, \tag{5.8}$$

where $C > 0$ is a constant depending on a, b, but not on h_k.

Let $\mathfrak{z}^{h_k} = \delta_x^4 \mathfrak{w}^{h_k}$. We normalize by setting

$$\widetilde{\mathfrak{w}}^{h_k} = \frac{\mathfrak{w}^{h_k}}{|\mathfrak{z}^{h_k}|_{h_k}}, \ \widetilde{\mathfrak{z}}^{h_k} = \frac{\mathfrak{z}^{h_k}}{|\mathfrak{z}^{h_k}|_{h_k}}.$$

Note in particular that $|\widetilde{\mathfrak{z}}^{h_k}|_h = 1$. Equation (5.3) can be rewritten as

$$\delta_x^4 \widetilde{\mathfrak{w}}^{h_k} + a\widetilde{\delta}_x^2 \widetilde{\mathfrak{w}}^{h_k} + b\widetilde{\mathfrak{w}}^{h_k} = \frac{\mathfrak{r}^{h_k}}{|\mathfrak{z}^{h_k}|_{h_k}}, \quad k = 1, 2, \dots. \tag{5.9}$$

By the above normalization the condition (4.3) is satisfied (with $\mathfrak{v}^{(k)}$ there corresponding to $\widetilde{\mathfrak{z}}^{h_k}$ here). Let $\widetilde{w}_{h_k}(x)$, $\widetilde{z}_{h_k}(x)$ be the piecewise-linear continuous functions corresponding, respectively, to $\widetilde{\mathfrak{w}}^{h_k}$, $\widetilde{\mathfrak{z}}^{h_k}$. We can invoke Corollary 4.3 and obtain that the following limit exists.

$$\lim_{k\to\infty} \widetilde{w}_{h_k}(x) = \widetilde{w}(x) \text{ in } C(\Omega), \tag{5.10}$$

The limit function $\widetilde{w}(x)$ is in $H^4(\Omega) \cap H_0^2(\Omega)$ and satisfies the equation

$$\Big(\frac{d}{dx}\Big)^4 \widetilde{w} + a\Big(\frac{d}{dx}\Big)^2 \widetilde{w} + b\widetilde{w} = 0. \tag{5.11}$$

In view of the Assumption (2.1) we must have

$$\widetilde{w} \equiv 0.$$

However , from (5.8) we get $1 \leq \frac{C}{|\delta_x^4 \mathfrak{w}^{h_k}|_{h_k}^2} + |\widetilde{\mathfrak{w}}^{h_k}|_{h_k}^2$. Owing to (5.6)) we conclude that for some $\eta > 0$,

$$|\widetilde{\mathfrak{w}}^{h_k}|_{h_k} > \eta, \quad k = 1, 2, \dots, \tag{5.12}$$

hence by (5.10) also

$$|\widetilde{w}|_{L^2(\Omega)} \geq \eta,$$

which is a contradiction. Thus, for some $N_0 > 1$,

$$|\delta_x^4 \mathfrak{w}^h|_h < C, \quad N_0 < N, \ h = \frac{1}{N}.$$

Finally, the other two estimates in (5.5) follow from the coercivity property (3.13).

□

5.2. **Error estimates by the general elliptic (energy) approach.** The discrete (finite difference) operators introduced in Section 3, acting on grid functions associated with smooth functions, should approximate the corresponding differential operators, as $h \to 0$. Obviously, the first requirement is the "consistency", namely, that the "truncation error" should vanish as $h \to 0$. However, we aim to derive "accuracy" estimates, measuring the difference between the discrete and continuous solutions, in an appropriate functional setting.

We first establish an error estimate in terms of the truncation error $\mathfrak{t}(h)$ involved in the discretization of the simplified equation (5.1). This is achieved as a straightforward application of the fundamental Theorem 5.2.

The truncation error results from replacing the continuous differential operators by their discrete analogs. We use a superscript "h" to indicate the dependence on the mesh size. Thus

$$\delta_x^4 u^{*,h} + a\widetilde{\delta}_x^2 u^{*,h} + bu^{*,h} = f^{*,h} + \mathfrak{t}(h). \tag{5.13}$$

Note that $\mathfrak{t}(h) \in l^2_{h,0}$.

Let $\mathfrak{g}^h \in l^2_{h,0}$ be the solution to (5.2).

The "error" grid function is defined as

$$\mathfrak{e}^h = u^{*,h} - \mathfrak{g}^h, \tag{5.14}$$

and subtracting (5.2) from (5.13) we obtain

$$\delta_x^4 \mathfrak{e}^h + a\widetilde{\delta}_x^2 \mathfrak{e}^h + b\mathfrak{e}^h = \mathfrak{t}(h). \tag{5.15}$$

The error estimate is then given in the following theorem.

Theorem 5.3. *[**Convergence by elliptic estimates**] The convergence of the discrete solution to the continuous solution is of order $\frac{3}{2}$, namely, $\mathfrak{e} = O(h^{\frac{3}{2}})$. The same rate applies also to the discrete derivatives up to fourth order.*

More explicitly, there exists a constant $C > 0$, depending only on f, and an integer $N_0 > 1$, such that

$$|\delta_x^4 \mathfrak{e}^h|_h + |\widetilde{\delta}_x^2 \mathfrak{e}^h|_h + |\mathfrak{e}^h|_h \le Ch^{\frac{3}{2}}, \quad N_0 < N, \ h = \frac{1}{N}. \tag{5.16}$$

Proof. From Theorem 5.2 we infer that there exists an integer $N_0 > 1$ and a constant (depending only on N_0) $C > 0$ such that

$$|\delta_x^4 \mathfrak{e}^h|_h + |\widetilde{\delta}_x^2 \mathfrak{e}^h|_h + |\mathfrak{e}^h|_h \le C|\mathfrak{t}(h)|_h, \quad N_0 < N, \ h = \frac{1}{N}. \tag{5.17}$$

To get a detailed estimate, we take a closer look at the truncation term $\mathfrak{t}(h)$. Due to the presence of a boundary (in contrast to the periodic case), the near-boundary points display a lower order of accuracy. In fact, we have by Taylor's expansion

$$(\delta_x^2 u^*)_j = \Big(\Big(\frac{d}{dx}\Big)^2 u\Big)^*_j + \frac{h^2}{12}(u^{(4)})^*_j + O(h^4) \quad 1 \le j \le N-1.$$

The derivative $(u^{(4)})^*_j$ can be replaced by $(\delta_x^4 u^*)_j$, with truncation error $O(h)$ for $j = 1, N-1$ and $O(h^4)$ for $2 \le j \le N-2$ [3, Proposition 10.8]. Thus in view of Equation (3.12) we obtain

$$\Big|\Big[\Big(\frac{d}{dx}\Big)^2 u\Big]^*_j - [\widetilde{\delta}_x^2 u^*]_j\Big| \le \begin{cases} Ch^3, & j = 1, N-1, \\ Ch^4, & 2 \le j \le N-2. \end{cases} \tag{5.18}$$

As for the fourth-order derivative, we have , using again [3, Proposition 10.8] (and the Simpson operator σ_x defined in (3.5)),

$$\left|\sigma_x\left[\left(u^{(4)}\right)^* - \delta_x^4 u^*\right]_j\right| \le \begin{cases} Ch, & j = 1, N-1, \\ Ch^4, & 2 \le j \le N-2. \end{cases} \tag{5.19}$$

From these two estimates we infer,

$$|\mathfrak{t}(h)|_h^2 \le Ch\Big[h^2 + Nh^8\Big] \le Ch^3, \tag{5.20}$$

where $C > 0$ depends only on f.

(Compare the proof of Proposition 10.8 in [3, Eq. (10.66)] for the pure biharmonic equation).

Inserting this estimate in (5.17) we get (5.16).

□

Remark 5.4. *Note that the theorem does not give us the "optimal" h^4 convergence. This is due to the presence of boundary conditions, as mentioned in Remark 5.1 above. Recall that in the "pure" case $a = b = 0$ we do have the optimal error estimate as in Claim 3.9.*

On the other hand, it gives us also estimates for the (discrete) derivatives of the error. In what follows we rely on these estimates in order to recover, in Theorem 5.7, an optimal (h^4) error estimate for $\mathfrak{e}$ satisfying (5.15), *but not for its derivatives.*

5.3. **Optimal error estimate.** The "sub-optimal" estimates (Remark 5.4) were a consequence of the loss of accuracy near the boundary (see (5.18) , (5.19)). The remedy to that fact is to apply the inverse operator $\left(\delta_x^4\right)^{-1}$, which retains optimal accuracy also near the boundary. This is what we do next.

The following proposition deals with the approximation of the second-order derivative.

Proposition 5.5. *For a smooth function $u(x)$, satisfying the homogeneous boundary conditions,*

$$\left(\delta_x^4\right)^{-1}\left[\left(\left(\frac{d}{dx}\right)^2 u\right)_j^* - \widetilde{\delta}_x^2 u_j^*\right] = O(h^4), \quad 1 \le j \le N-1. \tag{5.21}$$

Proof. Using Taylor's expansion

$$(\delta_x^2 u^*)_j = \left(\left(\frac{d}{dx}\right)^2 u\right)_j^* + \frac{h^2}{12}(u^{(4)})_j^* + O(h^4), \quad 1 \le j \le N-1,$$

so that acting with σ_x yields

$$(\sigma_x \delta_x^2 u^*)_j = \sigma_x\left[\left(\left(\frac{d}{dx}\right)^2 u\right)^*\right]_j + \frac{h^2}{12}\sigma_x[(u^{(4)})^*]_j + O(h^4), \quad 1 \le j \le N-1.$$

In view of the equality (3.12), it follows that,

$$\left(\sigma_x \widetilde{\delta}_x^2 u^*\right)_j = (\sigma_x \delta_x^2 u^*)_j - \frac{h^2}{12}(\sigma_x \delta_x^4 u^*)_j,$$

hence, for $1 \le j \le N-1$,

$$\left(\sigma_x \widetilde{\delta}_x^2 u^*\right)_j = \sigma_x\left[\left(\left(\frac{d}{dx}\right)^2 u\right)^*\right]_j + \frac{h^2}{12}\Big[\sigma_x[(u^{(4)})^* - \delta_x^4 u^*]\Big]_j + O(h^4). \tag{5.22}$$

We know that [3, Proposition 10.8], [10, Proposition 3],

$$\sigma_x[(u^{(4)})^* - \delta_x^4 u^*)]_j = \begin{cases} O(h^4), \ j = 2, ..., N-2, \\ O(h), \ j = 1, N-1. \end{cases} \quad . \tag{5.23}$$

Thus

$$R(u) := \sigma_x\Big[\widetilde{\delta}_x^2 u^* - \Big(\Big(\frac{d}{dx}\Big)^2 u\Big)^*\Big]_j = \begin{cases} O(h^4), \ j = 2, ..., N-2, \\ O(h^3), \ j = 1, N-1. \end{cases} \quad . \tag{5.24}$$

Now we can write

$$\begin{aligned} &\sigma_x^{-1}\Big(\delta_x^4\Big)^{-1}\Big[\Big(\Big(\frac{d}{dx}\Big)^2 u\Big)^* - \widetilde{\delta}_x^2 u^*\Big] \\ &\qquad = \Big(\sigma_x \delta_x^4 \sigma_x\Big)^{-1} R(u). \end{aligned} \tag{5.25}$$

Clearly the operators σ_x, σ_x^{-1} (see (3.5)) are uniformly (with respect to the mesh size h) bounded, so in order to prove the estimate (5.21) it suffices to estimate the right-hand side of (5.25). At this point we invoke the detailed structure of the matrix $(\widetilde{S})^{-1}$ associated with the operator $\Big(\sigma_x \delta_x^4 \sigma_x\Big)^{-1}$, see [3, Theorem 10.19], [10, Theorem 6]. In fact $\widetilde{S} = PSP$ in [3, Equation (10.111)] . The result we need is the following: the scales (in powers of h) of the elements of $(\widetilde{S})^{-1}$ are such that all the components of the vector $(\widetilde{S})^{-1}R(u)$, where $R(u)$ satisfies (5.24), are $O(h^4)$. This therefore concludes the proof. □

We now rewrite Equation (5.1) as

$$\Big(\frac{d}{dx}\Big)^{-4} f = u + a\Big(\frac{d}{dx}\Big)^{-4}\Big(\frac{d}{dx}\Big)^2 u + b\Big(\frac{d}{dx}\Big)^{-4} u, \quad x \in \Omega = [0, 1], \tag{5.26}$$

subject to the homogeneous boundary conditions

$$u(0) = u(1) = u'(0) = u'(1) = 0,$$

and Equation (5.2) as

$$(\delta_x^4)^{-1} f^{*,h} = \mathfrak{g}^h + a(\delta_x^4)^{-1}\widetilde{\delta}_x^2 \mathfrak{g}^h + b(\delta_x^4)^{-1}\mathfrak{g}^h, \quad h = \frac{1}{N}, \tag{5.27}$$

subject to the boundary conditions

$$\mathfrak{g}_0^h = (\mathfrak{g}_x^h)_0 = \mathfrak{g}_N^h = (\mathfrak{g}_x^h)_N = 0.$$

Now Claim 3.9 says that (for smooth functions) we can replace the continuous operator $\Big(\frac{d}{dx}\Big)^{-4}$ (evaluated at grid points) by the discrete operator $\Big(\delta_x^4\Big)^{-1}$ "at the expense" of an $O(h^4)$ error.

Thus, we conclude from (5.26) that

$$(\delta_x^4)^{-1} f^{*,h} = u^{*,h} + (\delta_x^4)^{-1}\Big[a\Big(\frac{d}{dx}\Big)^2 u + bu\Big]^{*,h} + O(h^4). \tag{5.28}$$

Given Proposition 5.5, we obtain from (5.28)

$$\begin{aligned} &(\delta_x^4)^{-1} f^{*,h} \\ &= u^{*,h} + a\Big(\delta_x^4\Big)^{-1}\widetilde{\delta}_x^2 u^{*,h} + b\Big(\delta_x^4\Big)^{-1} u^{*,h} + O(h^4). \end{aligned} \tag{5.29}$$

Subtracting (5.27) from (5.29) we obtain, with a constant $C > 0$ independent of h,

$$0 = \mathfrak{e}^h + a\Big(\delta_x^4\Big)^{-1}\widetilde{\delta}_x^2\mathfrak{e}^h + b\Big(\delta_x^4\Big)^{-1}\mathfrak{e}^h + \mathfrak{r}^h,\ |\mathfrak{r}^h|_h \le Ch^4. \tag{5.30}$$

The error term $\mathfrak{r}^h$ is now majorized by h^4 and our goal is to obtain a similar estimate for $\mathfrak{e}^h$ from Equation (5.30). Note that Equation (5.26) is not a differential equation, but rather a "pseudo-differential" one. Similarly Equation (5.27) is a "discrete pseudo-differential" equation. In seeking an estimate for $\mathfrak{e}^h$ from Equation (5.30) we shall therefore need a **pseudo-differential** version of the discrete elliptic Theorem 5.2. The result will depend on the "sub-optimal" (see Remark 5.4) estimates obtained in Theorem 5.3. While the estimates there were not of fourth-order, *they involved also the discrete derivatives of the error*. We shall incorporate these estimates (for $\mathfrak{e}^h$ and its derivatives) in the following proposition.

Proposition 5.6. *[**Fundamental discrete pseudo-differential estimates**] Let $\Big\{\mathfrak{v}^h \in l^2_{h,0},\ 0 < h < h_0\Big\}$ be a family of solutions (depending on the mesh-size parameter h) to the equation*

$$\mathfrak{v}^h + a\Big(\delta_x^4\Big)^{-1}\widetilde{\delta}_x^2\mathfrak{v}^h + b\Big(\delta_x^4\Big)^{-1}\mathfrak{v}^h = \mathfrak{r}^h. \tag{5.31}$$

Assume that $\mathfrak{v}_x^h \in l^2_{h,0}$, and that, for some $\beta > 0$ (independent of h)

$$|\delta_x^4\mathfrak{v}^h|_h \le \beta. \tag{5.32}$$

Then there exists an integer $N_0 > 1$ and a constant $C > 0$ (depending only on N_0) such that

$$|\mathfrak{v}^h|_h \le C|\mathfrak{r}^h|_h, \quad N_0 < N,\ h = \frac{1}{N}. \tag{5.33}$$

Proof. Suppose to the contrary that there exist sequences $h_k \to 0$ and $\Big\{\mathfrak{r}^{h_k} \in l^2_{h_k,0}\Big\}_{k=1}^{\infty}$ such that

$$\lim_{k\to\infty} |\mathfrak{r}^{h_k}|_{h_k} = 0 \tag{5.34}$$

while

$$|\mathfrak{v}^{h_k}|_{h_k} = 1, \quad k = 1, 2, \dots. \tag{5.35}$$

Let $\mathfrak{g}^{h_k} = (\delta_x^4)^{-1}\mathfrak{v}^{h_k}$, so that Equation (5.31) takes the form

$$\mathfrak{v}^{h_k} + a\Big(\delta_x^4\Big)^{-1}\widetilde{\delta}_x^2\mathfrak{v}^{h_k} + b\mathfrak{g}^{h_k} = \mathfrak{r}^{h_k}. \tag{5.36}$$

Note that the operators δ_x^4 and $\widetilde{\delta}_x^2$ do not commute, **and this is the reason we cannot invoke Theorem 5.2 at this point.**

By (5.35) $|\delta_x^4\mathfrak{g}^{h_k}|_{h_k} = |\mathfrak{v}^{h_k}|_{h_k} = 1$.

Let $\psi \in C_0^\infty(0,1)$ and set $\psi^{(4)} = \phi$. Let $\Big\{\phi^*_{h_k} \in l^2_{h_k,0},\ k = 1, 2, ...\Big\}$ be the sequence of grid functions corresponding to $\phi(x)$ (see (3.1)).

Taking the scalar product of the equality in (5.36) with $\phi^*_{h_k}$ and using the symmetry of the discrete operators we get

$$(\mathfrak{v}^{h_k}, \phi^*_{h_k})_{h_k} + a\Big(\mathfrak{v}^{h_k}, \widetilde{\delta}_x^2(\delta_x^4)^{-1}\phi^*_{h_k}\Big)_{h_k} + b(\mathfrak{g}^{h_k}, \phi^*_{h_k})_{h_k} = (\mathfrak{r}^{h_k}, \phi^*_{h_k})_{h_k}. \tag{5.37}$$

By assumption (5.34) the right-hand side in Equation (5.37) tends to zero as $k \to \infty$.

Denote $\mathfrak{w}^{h_k} = \widetilde{\delta}_x^2 (\delta_x^4)^{-1} \phi^*_{h_k}$.

We recall Definition 3.6 and introduce the continuous, piecewise linear functions $v_{h_k}(x),\ g_{h_k}(x),\ w_{h_k}(x),\ \phi_{h_k}(x)$ corresponding, respectively, to the grid functions $\mathfrak{v}^{h_k}$, $\mathfrak{g}^{h_k}$, $\mathfrak{w}^{h_k}$, $\phi^*_{h_k}$.

The discrete scalar products in (5.37) can be replaced by integrals of the corresponding functions, using the algebraic equality [3, Lemma 10.4]):

$$(\mathfrak{v}^{h_k}, \phi^*_{h_k})_{h_k} = \int_0^1 v_{h_k}(x)\phi_{h_k}(x)dx \tag{5.38}$$
$$+\frac{h_k}{6}\sum_{m=0}^{N_k-1}((\mathfrak{v}^{h_k})_{m+1} - (\mathfrak{v}^{h_k})_m)((\phi^*_{h_k})_{m+1} - (\phi^*_{h_k})_m), \quad N_k = \frac{1}{h_k},$$

and similarly for the other terms (compare (3.37)).

we therefore have (compare derivation of (3.38))

$$\lim_{k\to\infty} (v_{h_k}, \phi_{h_k})_{L^2(\Omega)} + a(v_{h_k}, w_{h_k})_{L^2(\Omega)} + b(g_{h_k}, \phi_{h_k})_{L^2(\Omega)} = 0, \tag{5.39}$$

Since $|\mathfrak{v}^{h_k}|_{h_k} = 1,\ k = 1, 2, ..$ we can invoke the compactness Theorem 3.7 and obtain a subsequence $\Big\{g_j(x) = g_{h_{k_j}}(x)\Big\}_{j=1}^{\infty}$ converging uniformly to a function $g(x) \in H^4(\Omega) \cap H^2_0(\Omega)$. Furthermore, the corresponding subsequence $\Big\{v_j(x) = v_{h_{k_j}}(x)\Big\}_{j=1}^{\infty}$ converges weakly to $v(x) = g^{(4)}(x)$. However, this convergence is in fact **uniform** in view of the assumption (5.32) (again using Theorem 3.7). We therefore have

$$v \in H^4(\Omega) \cap H^2_0(\Omega),$$

and the normalization assumption (5.35) entails

$$|v|_{L^2(\Omega)} = 1. \tag{5.40}$$

We can use Proposition 3.11, with the function g there replaced by ψ here, namely, $\lim\limits_{k\to\infty} w_{h_k}(x) = \psi''(x)$ in $C(\Omega)$. Passing to the limit (as $j \to \infty$) in Equation (5.39) we obtain,

$$(g^{(4)}, \phi)_{L^2(\Omega)} + a(g^{(4)}, \psi'')_{L^2(\Omega)} + b(g, \phi)_{L^2(\Omega)} = 0. \tag{5.41}$$

Since $\psi \in C_0^\infty(0, 1)$ the same is true for its derivatives and we can integrate twice by parts in the middle term, so that

$$(g^{(4)}, \psi'')_{L^2(\Omega)} = (g'', \phi)_{L^2(\Omega)},$$

and inserting this in (5.41)

$$(g^{(4)} + ag'' + bg, \phi)_{L^2(\Omega)} = 0. \tag{5.42}$$

Since $\psi^{(4)} = \phi$ it follows that

$$\Big((\frac{d}{dx})^4 (g^{(4)} + ag'' + bg), \psi\Big)_{L^2(\Omega)} = 0.$$

From the fact that ψ is a general test function we infer that

$$(\frac{d}{dx})^4 (g^{(4)} + ag'' + bg) = 0 \Rightarrow g^{(4)} + ag'' + bg = p(x),$$

where $p(x)$ is at most a cubic polynomial.

Recall that $v(x) = g^{(4)}(x)$ and $v \in H_0^2 \cap H^4$, so we can differentiate the last equation four times to get

$$v^{(4)} + av'' + bv = 0. \tag{5.43}$$

Assumption (2.1) implies $v = 0$. This is in contradiction to (5.40), thus proving (5.33). □

The optimal (fourth-order) estimate of the error is stated in the following theorem.

Theorem 5.7. ***[Fourth-order estimate of the error]*** *Consider Equation* (5.1) *and the corresponding finite difference scheme* (5.2)*.*

Let $\mathfrak{e}^h = u^{*,h} - \mathfrak{g}^h$ *be the error grid function* (5.14)*.*

Then we have an optimal estimate

$$|\mathfrak{e}^h|_h = O(h^4).$$

More explicitly, there exists a constant $C > 0$*, depending only on* f*, and an integer* $N_0 > 1$*, such that*

$$|\mathfrak{e}^h|_h \le Ch^4, \quad N_0 < N,\ h = \frac{1}{N}. \tag{5.44}$$

Proof. $\mathfrak{e}^h$ satisfies Equation (5.30). From Theorem 5.3 we know that it satisfies the condition (5.32), hence we can apply Proposition 5.6. Thus, from (5.30)

$$|\mathfrak{e}^h|_h \le C|\mathfrak{r}^h|_h \le Ch^4.$$

□

Remark 5.8. ***(small coefficients)*** *If the coefficients* a, b *are small then the optimal error estimate follows directly from the invertibility of Equation* (5.30)*, in view of the explicit bounds (independent of* h*) in Remark 3.5.*

The optimal estimate of Theorem 5.7 is in contrast to the estimates in Theorem 5.3, concerning the errors involved in comparing the derivatives of the exact solution to those of the discrete one. Invoking the coercivity property of the discrete biharmonic operator, we can actually improve these estimates as follows.

Corollary 5.9. *The Hermitian derivative* $\mathfrak{e}_x^h$ *and the second-order derivative* $\widetilde{\delta}_x^2\mathfrak{e}^h$ *of the error function are, respectively, of order* $O(h^{\frac{27}{8}})$ *and* $O(h^{\frac{11}{4}})$*. More explicitly, there exists a constant* $C > 0$*, independent of* h*, such that*

$$|\mathfrak{e}_x^h|_h \le Ch^{\frac{27}{8}}, \quad |\widetilde{\delta}_x^2\mathfrak{e}^h|_h \le Ch^{\frac{11}{4}}, \quad N_0 < N,\ h = \frac{1}{N}. \tag{5.45}$$

Proof. Applying the coercivity property (3.13) to $\mathfrak{e}^h$ and using the estimates (5.44) and (5.16) we get

$$|\widetilde{\delta}_x^2\mathfrak{e}^h|_h^2 \le C|\delta_x^4\mathfrak{e}^h|_h|\mathfrak{e}^h|_h \le Ch^{\frac{3}{2}}h^4,$$

hence indeed the $O(h^{\frac{11}{4}})$ for $\widetilde{\delta}_x^2\mathfrak{e}^h$.

We now use the coercivity property of the second-order derivative [3, Equation (9.34)] and the previous estimate to get

$$|\delta_x\mathfrak{e}^h|_h^2 \le C|\delta_x^2\mathfrak{e}^h|_h|\mathfrak{e}^h|_h \le Ch^{4+\frac{11}{4}} = Ch^{\frac{27}{4}},$$

from which we infer that $|\delta_x \mathfrak{e}^h|_h = O(h^{\frac{27}{8}})$. However by definition (3.9) we have $\mathfrak{e}_x^h = \sigma_x^{-1}\delta_x \mathfrak{e}^h$, and the operator σ_x^{-1} is uniformly (with respect to h) bounded [3, Equation (10.24)]. It follows that also $|\mathfrak{e}_x^h|_h = O(h^{\frac{27}{8}})$. □

6. NUMERICAL RESULTS

In this section we present numerical results of a representative set of test problems. The underlying equation is always (1.1), subject to the homogeneous boundary conditions (1.2). The scheme used is (4.1).

Our notation here is in accord with that employed in the previous sections, in particular Section 5. For the reader's convenience we recall the main features to be used here as follows.

For a given continuous function $v(x)$, $x \in [0,1]$, we denote by v^* (3.1) its associated grid function . When it is expedient to indicate explicitly the mesh size h, we use the notation $v^{*,h}$, as in (5.13).

$$v_j^{*,h} = v(x_j), \quad x_j = jh,\ 0 \le j \le N.$$

$\mathfrak{g}^h$ (4.1) is the discrete solution, approximating the analytic solution $u(x)$.

$\mathfrak{e}^h = u^{*,h} - \mathfrak{g}^h$ is the error grid function (5.14).

The discrete grid functions corresponding to the second-order and third-order derivatives are, respectively, $\widetilde{\delta_x^2}\mathfrak{g}^h$ (3.10) and $\delta_x^2\mathfrak{g}_x^h$.

The discrete norms $|\cdot|_h$ and $|\cdot|_\infty$ are defined, respectively, by (3.2) and (3.3) .

For linear operators $\mathcal{A} : l_h^2 \to l_h^2$ we use $|\mathcal{A}|_h$ to denote the operator norm.

Remark 6.1 (**Concerning errors for derivatives**). *In Corollary 5.9 we derived estimates for the derivatives of the error function $\mathfrak{e}^h$. These are the discrete derivatives, so that only values of the exact solution itself are used. In contrast, in our numerical test cases here we compare (discrete) derivatives of the calculated solution to the grid functions corresponding to the exact derivatives. Thus, the error for the first derivative is displayed here as $(\frac{d}{dx}u)^{*,h} - \mathfrak{g}_x^h$ and not $u_x^{*,h} - \mathfrak{g}_x^h$, the difference of the Hermitian derivatives.*

Indeed, comparing with derivatives of the exact solution seems to be a stricter criterion. However, due to the high order accuracy of the Hermitian derivative ($(\frac{d}{dx}u)^{*,h} - u_x^{*,h} = O(h^4)$ [3, Lemma 10.1]) the two estimates are compatible.

A similar observation is valid for higher-order derivatives as well.

Remark 6.2 (**Numerical efficiency**). *Even though the main purpose of the paper is to present a "discrete elliptic theory", resulting in a high-order compact scheme, the solution of the linear system (4.1) is quite efficient. In fact, it involves (that is the compactness of the scheme) the inversion of a three-diagonal matrix of size $N \times N$. In addition, another (simultaneous) inversion of the three-diagonal Simpson matrix σ_x (see (3.9)) is required for the connection of the unknown $\mathfrak{g}^h$ to its Hermitian derivative $\mathfrak{g}_x^h$. Thus the algorithm requires the inversion of two three-diagonal $N \times N$ matrices. The fact that the scheme possesses "optimal accuracy" enables us to use a very low N. This is demonstrated in the numerical examples hereafter.*

We display numerical results for three test cases.

- The first test case deals with the pure biharmonic equation. We give detailed results for the differences of all derivatives (up to third order).

- The second test case is an example of Equation (5.1), and shows fourth order accuracy for u, in agreement with Theorem 5.7. Fourth order accuracy is actually obtained not only for u but also for the derivative $u'(x)$, whereas Corollary 5.9 states only $O(h^{\frac{27}{8}})$ error estimate for the derivative .
- The third test case is a numerical example introduced in [10].
  The solution is highly oscillatory around the center of the interval $[0,1]$.

6.1. **A pure biharmonic problem.** The first test case corresponds to the pure biharmonic problem

$$\begin{cases} u^{(4)}(x) = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0, \quad u'(0) = u'(1) = 0, \end{cases} \tag{6.1}$$

where

$$f(x) = \frac{e^x}{2}\mathfrak{Re}\Big[1 - (1+4i\pi)^4 e^{4i\pi x}\Big], \tag{6.2}$$

with exact solution

$$u(x) = e^x \sin^2(2\pi x). \tag{6.3}$$

The numerical scheme is (5.2) which in this case reduces to

$$\delta_x^4 \mathfrak{g}^h = f^{*,h}. \tag{6.4}$$

In Table 1 we display the errors using both the l^∞ norm (3.3) and the l_h^2 norm (3.2) on a number of grids ranging from $N = 8$ points (very coarse grid) to $N = 64$ points.

The observed convergence rates are 4, 4, 4 and 2 for $u(x)$, $u'(x)$, $u''(x)$ and $u^{(3)}(x)$, respectively, better than claimed in Corollary 5.9.

In Table 1 we also present, for $u(x)$, $u'(x)$, $u''(x)$, the relative errors. They have a magnitude of 2%, 0.1%, 0.01% and 0.005% on the grids $N = 8, 16, 32$ and 64, respectively. The relative error for $u^{(3)}(x)$ is of 20%, 5%, 1% and 0.5% on the same grids.

In the last row of Table 1 we display the truncation error for the $4-$th order derivative

$$\tau_j = \delta_x^4(u^{*,h} - \mathfrak{g}^h)_j, \quad 1 \le j \le N-1. \tag{6.5}$$

In view of (6.4) this is just the difference between the discrete operator δ_x^4 acting on the grid function $u^{*,h}$ (the exact solution restricted to the grid) and the grid function corresponding to the exact right-hand side $f(x)$.

In accordance with the analysis in [3, Section 10.4], we obtain an asymptotic value of $O(h)$ in the maximum norm and of $O(h^{\frac{3}{2}})$ in the l_h^2 norm.

	$N=8$	Rate	$N=16$	Rate	$N=32$	Rate	$N=64$
$\lvert u^{*,h}-\mathfrak{g}^h\rvert_\infty$	5.8852(-2)	4.43	2.7340(-3)	4.09	1.6000(-4)	4.03	9.8219(-6)
$\lvert u^{*,h}-\mathfrak{g}^h\rvert_\infty/\lVert u\rVert_\infty$	2.76(-2)		1.28(-3)		7.51(-5)		4.61(-6)
$\lvert u^{*,h}-\mathfrak{g}^h\rvert_h$	3.1390(-2)	4.43	1.4604(-3)	4.11	8.4766(-5)	4.03	5.2006(-6)
$\lvert(u')^{*,h}-\mathfrak{g}_x^h\rvert_\infty$	3.5830(-1)	4.15	2.0183(-2)	4.01	1.2489(-3)	4.01	7.7252(-5)
$\lvert(u')^{*,h}-\mathfrak{g}_x^h\rvert_\infty/\lVert u'\rVert_\infty$	2.55(-2)		1.44(-3)		8.89(-5)		5.50(-6)
$\lvert(u')^{*,h}-\mathfrak{g}_x^h\rvert_h$	2.3440(-1)	4.21	1.2680(-2)	4.05	7.6410(-4)	4.01	4.7323(-5)
$\lvert(u'')^{*,h}-(\widetilde{\delta}_x^2\mathfrak{g}^h)\rvert_\infty$	4.8479(+0)	3.92	3.1931(-1)	4.03	1.9543(-2)	3.98	1.2345(-3)
$\lvert(u'')^{*,h}-(\widetilde{\delta}_x^2\mathfrak{g}^h)\rvert_\infty/\lVert u''\rVert_\infty$	2.26(-2)		1.49(-3)		9.13(-5)		5.77(-6)
$\lvert(u'')^{*,h}-\widetilde{\delta}_x^2\mathfrak{g}^h\rvert_h$	2.6941(+0)	4.08	1.5902(-1)	4.00	9.9617(-3)	3.99	6.2722(-4)
$\lvert((\frac{d}{dx})^3u)^{*,h}-(\delta_x^2\mathfrak{g}_x^h)\rvert_\infty$	4.7894(+2)	1.95	1.2391(2)	1.95	3.2148(1)	2.00	8.0205(0)
$\frac{\lvert((\frac{d}{dx})^3u)^{*,h}-(\delta_x^2\mathfrak{g}_x^h)\rvert_\infty}{\lVert u^{(3)}\rVert_\infty}$	1.95(-1)		5.04(-2)		1.31(-2)		3.26(-3)
$\lvert((\frac{d}{dx})^3u)^{*,h}-\delta_x^2\mathfrak{g}_x^h\rvert_h$	2.6552(+2)	1.99	6.6681(1)	2.00	1.6714(1)	2.00	4.1869(0)
$\lvert\tau\rvert_\infty$	1.2395(+3)	1.80	3.5509(+2)	1.10	1.6531(+2)	1.02	8.1351(+1)
$\lvert\tau\rvert_h$	4.9450(+2)	2.32	9.8824(+1)	1.61	3.2373(1)	1.52	1.1249(+1)

TABLE 1. Error levels and convergence rates for the test case (6.1)-(6.3). For each function $u(x)$, $u'(x)$, $u''(x)$ and $(\frac{d}{dx})^3u(x)$ the max error, relative max error and l^2 errors are given. The convergence rates are 4 for $u(x)$, 4 for $u'(x)$, 4 for $u''(x)$ and 2 for $(\frac{d}{dx})^3u(x)$. On the last two lines, the truncation error for $\delta_x^4u^{*,h}-f^{*,h}$ are displayed in max norm (convergence rate 1) and l^2 norm (convergence rate 3/2).

6.2. **A regular test case.** We consider Equation (5.1) with $a=1, b=1$:

$$\left\{\begin{array}{l}\left(\frac{d}{dx}\right)^4u+\left(\frac{d}{dx}\right)^2u+u=f,\quad x\in\Omega=[0,1],\\ \qquad u(0)=u(1)=0,\quad u'(0)=u'(1)=0\end{array}\right. \tag{6.6}$$

Let

$$f(x)=\frac{e^x}{2}\Big\{3-\Big[\Big(3-7(4\pi)^2+(4\pi)^4\Big)\cos(4\pi x)-\Big(8\pi(3-32\pi^2)\Big)\sin(4\pi x)\Big]\Big\}.$$

The exact solution $u(x)$ is readily seen to be

$$u(x)=e^x\sin^2(2\pi x). \tag{6.7}$$

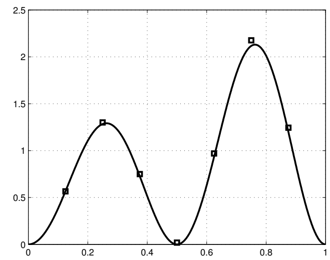
The scheme is (5.2) which in our case reduces to

$$(\delta_x^4\mathfrak{g}^h)_j+(\widetilde{\delta}_x^2\mathfrak{g}^h)_j+\mathfrak{g}_j^h=f_j^{*,h},\quad 1\le j\le N-1. \tag{6.8}$$

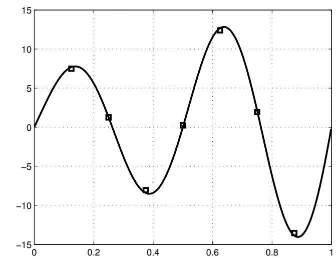
The functions $u(x)$, $u'(x)$, $u''(x)$, $(\frac{d}{dx})^3u(x)$ and $(\frac{d}{dx})^4u(x)$ are displayed in Fig. 1.

The values of the discrete solution $\mathfrak{g}^h$ and its subsequent discrete derivatives on the coarse grid $N=8$ are represented by squares. This very coarse grid corresponds to the minimally acceptable resolution with 5 points per wavelength.

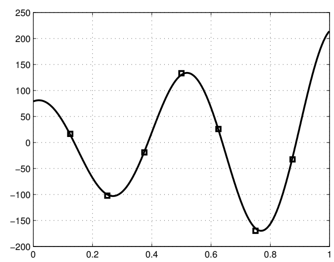
In Table 2 we give the error values in the l^∞ norm, the relative errors in the l^∞ norm and the errors in the l_h^2 norm for u, u', u'', $(\frac{d}{dx})^3u$ and $(\frac{d}{dx})^4u$.



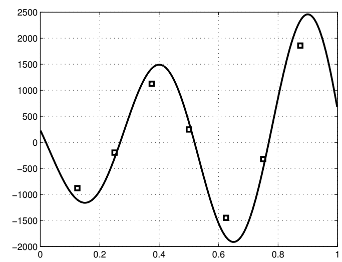
(A) Exact and calculated $u(x)$



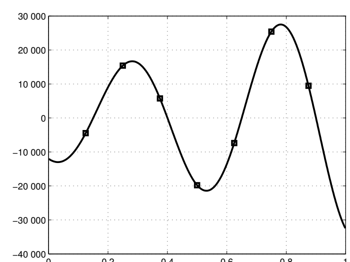
(B) Exact and calculated $u'(x)$



(C) Exact and calculated $u''(x)$



(D) Exact and calculated $(\frac{d}{dx})^3 u(x)$



(E) Exact and calculated $(\frac{d}{dx})^4 u(x)$

FIGURE 1. Exact (solid curve) and calculated solution (black squares) using a 7 point grid ($N = 8$), for equation (6.6). The exact solution is given in (6.7)) and the calculated solution is solution of (6.8). The magnitude of $u(x)$, $u'(x)$, $u''(x)$, $(\frac{d}{dx})^3 u(x)$ and $(\frac{d}{dx})^4 u(x)$ is accurately calculated even on the very coarse grid with parameter $N = 8$.

The excellent accuracy is clearly observed for $u(x)$, $u'(x)$, $u''(x)$ and $(\frac{d}{dx})^4 u(x)$.

	$N=8$	Rate	$N=16$	Rate	$N=32$	Rate	$N=64$
$\|u^{*,h}-\mathfrak{g}^h\|_\infty$	5.9484(-2)	4.43	2.7681(-3)	4.09	1.6199(-4)	4.03	9.9484(-6)
$\|u^{*,h}-\mathfrak{g}^h\|_\infty/\|u\|_\infty$	2.79(-2)		1.30(-3)		7.60(-5)		4.67(-6)
$\|u^{*,h}-\mathfrak{g}^h\|_h$	3.1726(-2)	4.42	1.4784(-3)	4.11	8.5855(-5)	4.03	5.2681(-6)
$\|(u')^{*,h}-\mathfrak{g}_x^h\|_\infty$	3.5792(-1)	4.15	2.0155(-2)	4.02	1.2465(-3)	4.02	7.7101(-5)
$\|(u')^{*,h}-\mathfrak{g}_x^h\|_\infty/\|u'\|_\infty$	2.55(-2)		1.43(-3)		8.87(-5)		5.49(-6)
$\|(u')^{*,h}-\mathfrak{g}_x^h\|_h$	2.3329(-1)	4.21	1.2609(-2)	4.05	7.5974(-4)	4.01	4.7051(-5)
$\|(u'')^{*,h}-\widetilde{\delta}_x^2\mathfrak{g}^h\|_\infty$	4.8838(+0)	3.93	3.2109(-1)	4.03	1.9667(-2)	3.99	1.2419(-3)
$\|(u'')^{*,h}-\widetilde{\delta}_x^2\mathfrak{g}^h\|_\infty/\|u''\|_\infty$	2.28(-2)		2.50(-3)		9.19(-5)		5.80(-6)
$\|(u'')^{*,h}-\widetilde{\delta}_x^2\mathfrak{g}^h\|_h$	2.7085(+0)	4.08	1.5991(-1)	4.00	1.0020(-2)	3.99	6.3095(-4)
$\|((\frac{d}{dx})^3u)^{*,h}-(\delta_x^2\mathfrak{g}_x^h)\|_\infty$	4.7848(+2)	1.95	1.2388(+2)	1.95	3.2146(+1)	2.00	8.0204(+0)
$\frac{\|((\frac{d}{dx})^3u)^{*,h}-(\delta_x^2\mathfrak{g}_x^h)\|_\infty}{\|(\frac{d}{dx})^3u\|_\infty}$	1.95(-1)		5.04(-2)		1.31(-2)		3.26(-3)
$\|((\frac{d}{dx})^3u)^{*,h}-(\delta_x^2\mathfrak{g}_x^h)\|_h$	2.6533(+2)	1.99	6.6667(+1)	2.00	1.6713(+1)	2.00	4.1668(+0)
$\|((\frac{d}{dx})^4u)^{*,h}-(\delta_x^4\mathfrak{g}^h)\|_\infty$	4.8245(+0)	3.92	3.1840(-1)	4.03	1.9505(-2)	3.98	1.23332(-3)
$\frac{\|((\frac{d}{dx})^4u)^{*,h}-(\delta_x^4\mathfrak{g}^h)\|_\infty}{\|(\frac{d}{dx})^4u\|_\infty}$	1.49(-4)		9.82(-6)		6.01(-7)		3.80(-8)
$\|((\frac{d}{dx})^4u)^{*,h}-(\delta_x^4\mathfrak{g}^h)\|_h$	2.6901(+0)	4.08	1.5916(-1)	4.00	9.9782(-3)	3.99	6.2845(-4)
$\|\tau\|_\infty$	1.2430(+3)	1.81	3.5504(+2)	1.10	1.6530(+2)	1.02	8.1350(+1)
$\|\tau\|_h$	4.9663(+2)	2.33	9.8831(+1)	1.61	3.2372(+1)	1.52	1.1249(+1)

Table 2. Error levels and convergence rates for the test case (6.6)-(6.7). For each function $u(x)$, $u'(x)$, $u''(x)$, $(\frac{d}{dx})^3u(x)$ and $(\frac{d}{dx})^4u(x)$, the max error, relative max error and l^2 errors are given. The convergence rates are 4 for $u(x)$, 4 for $u'(x)$, 4 for $u''(x)$, 2 for $(\frac{d}{dx})^3u(x)$ and 4 for $(\frac{d}{dx})^4u(x)$. On the last two lines, the truncation error for $\delta_x^4u^{*,h}-f^{*,h}$ are displayed in max norm (convergence rate 1) and l^2 norm (convergence rate 3/2).

6.3. **Oscillating test case.** Here we consider the full Equation (1.1):

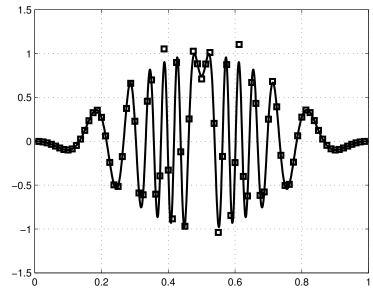
$$(6.9)\quad \begin{cases} \left(\frac{d}{dx}\right)^4 u + A(x)\left(\frac{d}{dx}\right)^2 u + A'(x)\left(\frac{d}{dx}\right)u + B(x)u(x) = f(x), \quad 0<x<1, \\ u(0)=u(1)=0, \quad u'(0)=u'(1)=0 \end{cases}$$

The functions $A(x)$ and $B(x)$ are taken as oscillatory (but regular) functions, defined by:

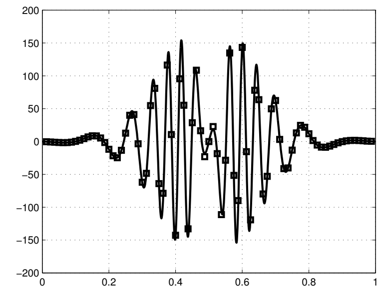
$$(6.10)\quad \begin{cases} A(x) = C_A(1+0.5\sin(40\pi x)); A'(x) = 20C_A\pi\cos(40\pi x); \\ B(x) = C_B\sin(40\pi x). \end{cases}$$

The function $u(x)$ is in this case [10]:

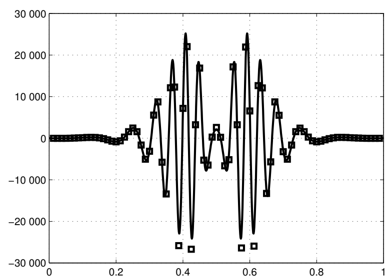
$$(6.11)\quad u(x) = p(x)/(\sin(q(x))+\varepsilon)$$



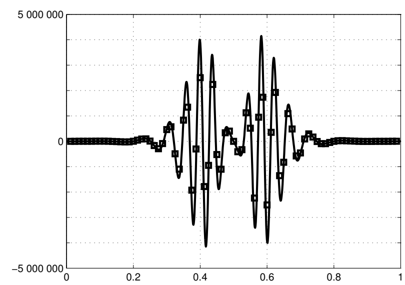
(A) Exact and calculated $u(x)$



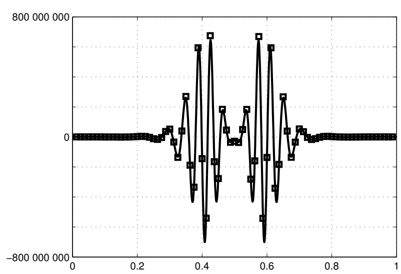
(B) Exact and calculated $u'(x)$



(C) Exact and calculated $u''(x)$



(D) Exact and calculated $(\frac{d}{dx})^3 u(x)$



(E) Exact and calculated $(\frac{d}{dx})^4 u(x)$

FIGURE 2. Exact (solid curve) and calculated solution (black squares) using a 80 point coarse grid ($N = 80$), for equation (6.9). The exact solution is given in (6.11))-(6.12)). Despite the highly oscillatory behavior of $u(x)$ and its derivatives, the magnitude of $u(x)$, $u'(x)$, $u''(x)$, $(\frac{d}{dx})^3 u(x)$ and $(\frac{d}{dx})^4 u(x)$ is very accurately captured even on the very coarse resolution of the $N = 80$ grid.

	$N=64$	Rate	$N=128$	Rate	$N=256$	Rate	$N=512$
$\lvert u^{*,h}-\mathfrak{g}^h\rvert_\infty$	8.3916(-1)	5.03	2.5677(-2)	4.08	1.5147(-3)	4.00	9.4041(-5)
$\lvert u^{*,h}-\mathfrak{g}^h\rvert_\infty/\lVert u\rVert_\infty$	8.4228(-1)		2.5777(-2)		1.5204(-3)		9.4390(-5)
$\lvert u^{*,h}-\mathfrak{g}^h\rvert_h$	1.7924(-1)	4.82	6.3341(-3)	4.11	3.6767(-4)	4.02	2.2604(-5)
$\lvert (u')^{*,h}-\mathfrak{g}_x^h\rvert_\infty\rvert$	3.9828(+1)	4.25	2.0891(+0)	4.17	1.1605(-1)	4.04	7.0551(-3)
$\lvert (u')^{*,h}-\mathfrak{g}_x^h\rvert_\infty/\lVert u'\rVert_\infty$	2.5858(-1)		1.3563(-2)		7.5342(-4)		4.5804(-5)
$\lvert (u')^{*,h}-\mathfrak{g}_x^h\rvert_h$	1.0114(+1)	4.43	4.6866(-1)	4.13	2.6838(-2)	4.03	1.6451(-3)
$\lvert (u'')^{*,h}-\widetilde{\delta}_x^2\mathfrak{g}^h\rvert_\infty$	1.4971(+4)	4.70	5.7429(+2)	4.04	3.4817(+1)	3.96	2.2349(+0)
$\frac{\lvert (u'')^{*,h}-\widetilde{\delta}_x^2\mathfrak{g}^h\rvert_\infty}{\lVert u''\rVert_\infty}$	5.9422(-1)		2.2794(-2)		1.3820(-3)		8.8707(-5)
$\lvert (u'')^{*,h}-\widetilde{\delta}_x^2\mathfrak{g}^h\rvert_h$	3.9786(+3)	4.85	1.3824(+2)	4.09	8.1374(+0)	4.02	5.0119(-1)
$\lvert (u^{(3)})^{*,h}-\delta_x^2\mathfrak{g}_x^h\rvert_\infty$	1.6790(+6)	1.60	5.5408(+5)	1.90	1.4872(+5)	1.99	3.7450(+4)
$\frac{\lvert (u^{(3)})^{*,h}-\delta_x^2\mathfrak{g}_x^h\rvert_\infty}{\lVert u^{(3)}\rVert_\infty}$	4.0468(-1)		1.3355(-1)		3.5845(-2)		9.0266(-3)
$\lvert (u^{(3)})^{*,h}-\delta_x^2\mathfrak{g}_x^h\rvert_h$	5.2628(+5)	1.80	1.5138(+5)	1.99	3.8188(+4)	2.00	9.57779(+3)
$\lvert (u^{(4)})^{*,h}-\delta_x^4\mathfrak{g}^h\rvert_\infty$	1.4720(+8)	4.65	5.8746(+6)	4.02	3.61729(+5)	3.95	2.3428(+4)
$\frac{\lvert (u^{(4)})^{*,h}-\delta_x^4\mathfrak{g}^h\rvert_\infty}{\lVert u^{(4)}\rVert_\infty}$	2.0996(-1)		8.3792(-3)		5.2094(-4)		3.3416(-5)
$\lvert (u^{(4)})^{*,h}-\delta_x^4\mathfrak{g}^h\rvert_h$	3.9528(+7)	4.86	1.3573(+6)	4.08	8.0024(+4)	4.02	4.9385(+3)
$\lvert \tau\rvert_\infty$	3.2156(+8)	4.86	1.10859(+7)	4.18	6.1042(+5)	4.01	3.7991(+4)
$\lvert \tau_j\rvert_h$	9.6786(+7)	5.19	2.6503(+6)	4.28	1.3638(+5)	4.07	8.1440(+3)

TABLE 3. Error levels and convergence rates for the test case (6.9)-(6.10). For each function $u(x)$, $u'(x)$, $u''(x)$, $(\frac{d}{dx})^3u(x)$ and $(\frac{d}{dx})^4u(x)$, the max error, relative max error and l^2 errors are given. The convergence rates are 4 for $u(x)$, 4 for $u'(x)$, 4 for $u''(x)$, 2 for $(\frac{d}{dx})^3u(x)$ and 4 for $(\frac{d}{dx})^4u(x)$. On the last two lines, the truncation error for $\delta_x^4u^{*,h}-f^{*,h}$ are displayed in max norm (convergence rate 4) and l^2 norm (convergence rate 4).

with

$$p(x)=x^2(1-x)^2,\quad q(x)=(x-1/2)^2,\quad \varepsilon>0 \tag{6.12}$$

The parameter $\varepsilon=0.025$ serves for monitoring the oscillations frequency. The source term $f(x)$ is obtained by applying Equation (6.9) to the function (6.11).

The scaling constants C_A and C_B are chosen to ensure that the magnitudes of the various terms in (6.9) are roughly equal.

The values are $C_A=10^4$ and $C_B=10^8$.

Taking into account the frequency of the oscillations of $A(x)$ and $B(x)$ in (6.10), a plausible stencil of 5 points per wavelength gives a mesh size of $h=1/80$. This resolution is therefore a lower bound for a computational grid.

The numerical scheme is now the full scheme (4.1).

In Fig. 2 we plot (as solid lines) the graphs of the exact solution and its derivatives, and indicate the corresponding discrete solutions using the coarse grid ($N = 80$, i.e. $h = 1/80$.) Even at this low resolution, all the five functions $u(x)$, $u(x)$, $u''(x)$, $(\frac{d}{dx})^3 u(x)$ and $(\frac{d}{dx})^4 u(x)$ are very well approximated. This is particularly true for the functions $u'(x)$ and $(\frac{d}{dx})^4 u(x)$. In Table 3 we display the errors, convergence rates, and relative errors for the grid functions corresponding to $u(x)$, $u(x)$, $u''(x)$, $u^{(3)}(x)$ and $u^{(4)}(x)$ compared, respectively, to their discrete analogs $\mathfrak{g}^h$, $\mathfrak{g}^h_x$, $\widetilde{\delta}^2_x \mathfrak{g}^h$, $\delta^2_x \mathfrak{g}^h_x$, $\delta^4_x \mathfrak{g}^h$.

Observe that the truncation errors (6.5)

$$\tau_j = \delta^4_x (u^{*,h} - \mathfrak{g}^h)_j, \quad 1 \le j \le N-1, \tag{6.13}$$

are of order 4, better than what could be inferred from Corollary 5.9. This is typical of a periodic behavior, due to the fact that all derivatives almost vanish near the boundary. See Remark 5.1. Fig. 3 shows the convergence rates for the discrete approximations to the functions $u(x)$, $u'(x)$, $u''(x)$, $(\frac{d}{dx})^3 u(x)$ and $(\frac{d}{dx})^4 u(x)$, in terms of decreasing h.

Fourth order convergence is observed for $u(x)$, $u'(x)$, $u''(x)$ and $(\frac{d}{dx})^4 u(x)$. Second order convergence is obtained for $(\frac{d}{dx})^3 u(x)$.

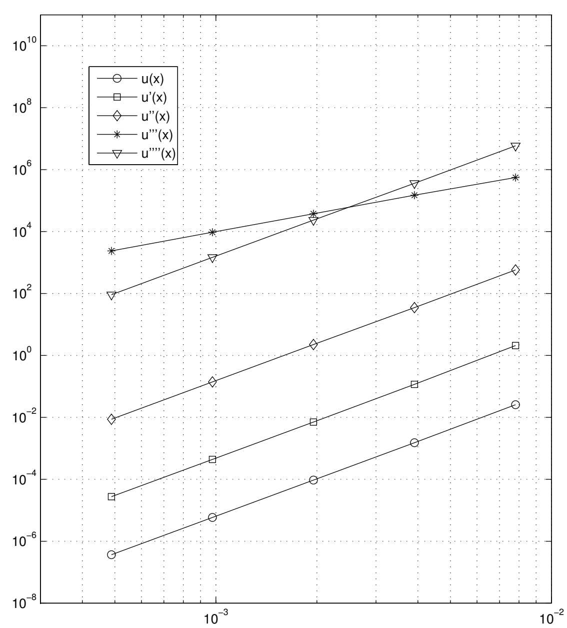


FIGURE 3. Convergence rate for the test case (6.9)-(6.10)). A series of 5 grids with $N = 128, 256, 512, 1024$ and 2048 is used. Fourth order is obtained for u, u', u'', $(\frac{d}{dx})^4 u$. Second order is obtained for $(\frac{d}{dx})^3 u$.

References

[1] A. A. Abushama and B. Bialecki, Modified nodal cubic spline collocation for biharmonic equations, Numer.Algor. **43** (2006), 331–353.

[2] M. Ben-Artzi, J.-P. Croisille and D. Fishelov, Convergence of a compact scheme for the pure streamfunction formulation of the unsteady Navier-Stokes system, SIAM J. Numer. Anal. **44** (2006), 1997–2024.

[3] M. Ben-Artzi, J.-P. Croisille and D. Fishelov, *"Navier-Stokes Equations in Planar Domains"*, Imperial College Press, 2013.

[4] A. Boumenir, Sampling for the fourth-order Sturm-Liouville differential operator, J. Math. Anal. Appl. **278** (2003), 542–550.

[5] S.C. Brenner and R. Scott, *"The Mathematical Theory of Finite Element Methods "*, Springer-Verlag (2012).

[6] B.M. Brown, E.B. Davies, P.K. Jimack and M.D. Mihajlović, A numerical investigation of the solution of a class of fourth-order eigenvalue problems, Proc. R. Soc. Lond. A **456** (2000), 1505–1521.

[7] E. A. Coddington and N. Levinson, *"Theory of Ordinary Differential Equations"*, McGraw Hill, 1955.

[8] E. B. Davies, *"Spectral Theory and Differential Operators"*, Cambridge University Press, 1995.

[9] L. C. Evans, *"Partial Differential Equations"*, American Mathematical Society 1998.

[10] D. Fishelov, M. Ben-Artzi and J.-P. Croisille, Recent advances in the study of a fourth-order compact scheme for the one-dimensional biharmonic equation, J. Sci. Comput. **53** (2012), 55–79.

[11] A. Friedman, *" Partial Differential Equations"*, Holt, Reinhart and Winston, 1969.

[12] L. Greenberg and M. Marletta, Numerical methods for higher order Sturm-Liouville problems, J. Comput. Applied Math. **125** (2000), 367–383.

[13] Y. Gupta and P.K. Srivastava, A computational method for solving two point boundary value problems of order four, Inter. J. Comp. Tech. Appl. **2** (2011),1426–1431.

[14] T.Y. Hou and B.T.R. Wetton, Convergence of a finite difference scheme for the Navier-Sokes equations using vorticity boundary conditions, SIAM J. Numer. Analysis **29** (1992), 615–639.

[15] A. Iserles, *"A First Course in the Numerical Analysis of Differential Equations"*, Cambridge University Press, 2008.

[16] B.S. Jovanović and E. Suli, *"Analysis of Finite Difference Schemes for Linear Partial Differential Equations"*, Springer, 2014.

[17] P. Kalyani, A.S. M. Rao and P.S. R. C. Rao, Numerical solution of fourth order boundary value problem using sixth degree spline functions, J. of Physics: Conference Seies **662** (2015), 1–5.

[18] K.N.S. Kasi Viswanadham, P. Murali Krishna and R.S. Koneru, Numerical solution of fourth order boundary value problems by Galerkin method with quintic B-splines, Inter. J. Nonlinear Science **10** (2010),222–230.

[19] A.G.C.P.Ramos and A. Iserles, Numerical solution of Sturm-Liouville problems via Fer streamers , Numer.Math. **131** (2015),541–565.

[20] J. Rashidinia and A. Golbabaee, Convergence of numerical solution of a fourth-order boundary value problem, Applied Math. Comput. **171** (2005), 1296–1305.

[21] A. Rattana and C. Böckmann, Matrix methods for computing eigenvalues of Sturm-Liouville problems of order four, J. Comp. and Applied Math. **249** (2013), 144–156.

[22] M. Reed and B. Simon, *"Methods of Modern Mathematical Physics, Vol. 4"* , Academic Press, 1978.

[23] S. S. Siddiki and G. Akram, Quintic spline solutions of fourth order boundary value problems, International J. Numer. Anal. Modeling **5** (2008), 101–111.

[24] A. Spence, On the convergence of the Nyström method for the integral equation eigenvalue problem, Numer. Math. **25** (1975), 57–66.

[25] R.A. Usmani, Discrete variable methods for a boundary value problem with engineering applications, Math. Comp. **32** (1978), 1087–1096.

[26] C. Wieners, Bounds for the N lowest eigenvalues of fourth-order boundary value problems, Computing **59** (1997), 29–41.

Matania Ben-Artzi: Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel
E-mail address: mbartzi@math.huji.ac.il

Jean-Pierre Croisille: Department of Mathematics, IECL, UMR 7502, Univ. de Lorraine, Metz 57045, France
E-mail address: jean-pierre.croisille@univ-lorraine.fr

Dalia Fishelov: Afeka - Tel-Aviv Academic College of Engineering, 218 Bnei-Efraim St., Tel-Aviv 69107, Israel
E-mail address: daliaf@afeka.ac.il

Ron Katzir:Applied Materials Israel Ltd., 9 Oppenheimer Street, Rehovot 76705, Israel
E-mail address: ron.katzir@gmail.com