Finite Volume Box Schemes

Jean-Pierre CROISILLE
Département de Mathématiques
Université de Metz
F-57045 Metz Cedex 01
croisil@poncelet.univ-metz.fr

ABSTRACT: *We present the numerical analysis on the Poisson problem of a mixed Petrov-Galerkin Finite Volume scheme for equations in divergence form* $\operatorname{div} \varphi(u, \nabla u) = f$*, which has been introduced in [CoC 98]. As the original box scheme of Keller, this scheme uses face centered degrees of freedom for the primal unknown* u *and for the flux* φ*. The underlying Finite Element spaces are the non-conforming space of Crouzeix-Raviart for the primal unknown and the div-conforming space of Raviart-Thomas for the flux. Optimal order error estimates are derived for the Poisson problem.*

Key Words: Finite-Volume method, Box Scheme, Box Method, Mixed Method.

1. Introduction

The name of "box-scheme" is a generic denomination for several numerical schemes of different origins. It has been introduced primitively by H.B. Keller in the '70 on the 1-D heat equation, [Ke 71]. Generally speaking, the discrete equations are defined in a box-scheme from some kind of averages of the continuous equations on "boxes". Therefore, they are conservative schemes, i.e. schemes which guarantee, for equations in divergence form, an exact conservation of the flux at the level of the box. At least two variants of box-schemes are known in the litterature. The first one has been introduced in the '80 in compressible Computational Fluid Dynamics. As in Keller's scheme, the basic idea is that locating the degrees of freedom at the center of the faces instead at the center of the cells, could be more interesting for an accurate evaluation of conservative fluxes, [CDH 83], [CM 86], [Co 92]. Note that this kind of schemes has received much less attention in the CFD Finite Volume communauty than the cell-centered Finite Volume schemes. The second kind of box-schemes is known under the name of “box-method” or “finite volume element method”. For the Laplace equation $-\Delta u = f$, it consists of an approximation of u in a finite-element P^1 or Q^1-space. The discrete equations are defined by averaging the equation on a dual box surroundind each vertex of the mesh, [BR 87], [Ha 89], [CMM 91], [TS 93]. In this kind of scheme, two meshes are used. The primal one as support of the trial functions, and the dual one for designing the boxes for the dicrete conservative equations. This design is in fact similar to the one of the cell-vertex Finite Volume method in CFD, [FS 89]. Concerning the numerical analysis of Finite Volume cell-centered methods, beside the exhaustive direct analysis of [EGH 97], there is by now an attempt in the FEM communauty for interpreting mixed Finite Elements methods as Finite Volume methods, [BMO 96], [Du 97], [YMAC 99]. The scheme presented here can be more or less attached to this kind of study.

2. The Finite Volume Box Scheme

In [CoC 98], we introduced a new kind of box-schemes for equations in divergence form. As in box-schemes with finite-differencing interpretation, [Ke 71], [Co 92], the degrees of freedom are located at the center of the faces of the mesh. Nevertheless, the discretization is interpreted here as a Finite Element approximation, allowing to use Finite Element theory for the numerical analysis. Let us consider the 2D Poisson problem in mixed form with Dirichlet homogeneous boundary conditions

$$\left\{ \begin{array}{c} \operatorname{div} p + f = 0 \;\text{ in } \Omega \\ p - \nabla u = 0 \;\text{ in } \Omega \\ u = 0 \;\text{ onto } \partial\Omega \end{array} \right. \tag{1}$$

Suppose given a triangulation $\mathcal{T}_h$ of the domain $\Omega \subset \mathbb{R}^2$ by triangles K. The number of triangles is NE. The number of internal edges, boundary edges are NA_i, NA_b and the total number of edges is $NA = NA_i + NA_b$. The Finite Element spaces that are used are

u-space: The non-conforming Crouzeix-Raviart space with homegeneous boundary conditions $V_h = P^1_{nc,0}$, equipped with the mesh dependent norm

$$\|u_h\|_h = (\sum_{K \in \mathcal{T}_h} |\nabla u_h|^2_{0,K})^{1/2}$$

p-space: The div-conforming Raviart-Thomas space of least order $Q_h = RT^0$, equipped with the continuous norm

$$\|p_h\|_{\mathrm{div}} = (|p_h|^2_{0,\Omega} + |\mathrm{div}\, p_h|^2_{0,\Omega})^{1/2}$$

Recall that these spaces are

$$V_h = \{v_h / \forall\ K \in \mathcal{T}_h,\ \ v_h|_K \in P^1(K), v_h \text{ is continuous at the middle of each edge}, v_h = 0 \text{ at the middle of each edge on } \partial\Omega\}$$

$$Q_h = \{q_h(x,y) \in H_{\mathrm{div}}(\Omega)\ /\ \forall\ K \in \mathcal{T}_h,\ q_h(x,y)|_K \in (P^0(K))^2 + P^0(K) \begin{bmatrix} x \\ y \end{bmatrix}\}$$

The scheme reads: find $(u_h, p_h) \in V_h \times Q_h$ such that

$$\left\{ \begin{array}{cc} \langle \mathrm{div}\, p_h + f, \mathbb{1}_K \rangle = 0 & \forall\ K \in \mathcal{T}_h \\ \langle p_h - \nabla u_h, \mathbb{1}_K \rangle = 0 & \forall\ K \in \mathcal{T}_h \\ u_h = 0 & \text{on } \partial\Omega \end{array} \right. \tag{2}$$

In (2), the number of unknowns is $2NA$, since the global degrees of freedom for u_h, p_h are scalars located at the center of the faces of the mesh. The number of equations is clearly $3NE + NA_b$. A simple count of the faces (the edges) proves that in fact we have

$$3NE + NA_b = 2NA. \tag{3}$$

Let us mention that coupling these two spaces is not standard in the mixed finite element methods, because they do not verify the Babuska-Brezzi condition, [Ba 71], [Bre 74].

3. Numerical Analysis

3.1. *Reformulation as a mixed Petrov-Galerkin method*

We consider the following mixed formulation of problem (1): find $(u, p) \in H^1_0 \times H_{\mathrm{div}}$ such that for any $(v, q) \in L^2 \times (L^2)^2$

$$B[(u,p);(v,q)] = (p,q) + (\mathrm{div}\, p, v) - (\nabla u, q) = -(f, v) \tag{4}$$

or equivalently

$$\left\{\begin{array}{ll}(\operatorname{div} p+f, v)=0 & \forall v \in L^2 \\ (p-\nabla u, q_)=0 & \forall q \in (L^2)^2\end{array}\right. \tag{5}$$

Applying the general Babuska theorem [Ba 71] onto mixed formulation we find easily that (5) is a well posed problem, whose solution is $(u, \nabla u)$, $u \in H_0^1 \cap H^2$ being the unique solution of the original problem (1). The scheme (2) appears now as a Petrov-Galerkin non conforming approximation of (5). Calling P^0 the space of constant functions in each triangle, it can be rewritten: find $(u_h, p_h) \in P^1_{nc,0} \times RT^0$ such that for any $(v_h, q_h) \in P^0 \times (P^0)^2$

$$(p_h, q_h) + (\operatorname{div} p_h, v_h) - \sum_{K \in \mathcal{T}_h} (\nabla u_h, q_h) = -(f, v_h) \tag{6}$$

or equivalently

$$\left\{\begin{array}{ll}(\operatorname{div} p_h+f, v_h)_{0,\Omega}=0 & \forall v_h \in P^0 \\ \sum_K (p_h-\nabla u_h, q_h)_{0,K}=0 & \forall q_h \in (P^0)^2\end{array}\right. \tag{7}$$

Applying now the theory for mixed Petrov-Galerkin approximations, [Ni 82], [BCM 88], [BMO 96], [Cr 99], we get the following result:

Theorem 1 *(i) The scheme (7) does possess an unique solution* $(u_h, p_h) \in P^1_{nc,0} \times RT^0$ *verifying*

$$\|u_h\|_h + \|p_h\|_{\operatorname{div}} \leq C\, |f|_{0,\Omega} \tag{8}$$

(ii) We have the error estimate

$$\|u - u_h\|_h + \|p - p_h\|_{\operatorname{div}} \leq Ch\, |f|_{0,\Omega} \tag{9}$$

3.2. *Dual scheme*

Another mixed form of the Poisson problem linked with the bilinear form B is dual from (1): find $(v, q) \in L^2 \times (L^2)^2$ such that for any $(u, p) \in H_0^1 \times H_{\operatorname{div}}$

$$B[(u,p);(v,q)] = -(f,u) \tag{10}$$

or equivalently

$$\left\{\begin{array}{ll}-(\nabla u, q)=-(f, u) & \forall u \in H_0^1 \\ (p, q)+(\operatorname{div} p, v)=0 & \forall p \in H_{\operatorname{div}}\end{array}\right. \tag{11}$$

Again by the Babuska theorem, problem (11) does possess an unique solution $(v,q) = (u, \nabla u)$. The corresponding Petrov-Galerkin scheme is: find $(v_h, q_h) \in P^0 \times (P^0)^2$ such that

$$\left\{ \begin{array}{ll} -(\nabla u_h, q_h) = -(f, u_h) & \forall u_h \in P^1_{nc,0} \\ (p_h, q_h) + (\operatorname{div} p_h, v_h) = 0 & \forall p_h \in RT^0 \end{array} \right. \tag{12}$$

As precedingly, we get the following result

Theorem 2 *(i) The dual scheme (12) does have an unique solution* $(v_h, q_h) \in P^0 \times (P^0)^2$ *verifying*

$$|v_h|_{0,\Omega} + |q_h|_{0,\Omega} \leq C\,|f|_{0,\Omega} \tag{13}$$

(ii) We have the error estimate

$$|v - v_h|_{0,\Omega} + |q - q_h|_{0,\Omega} \leq Ch\,|f|_{0,\Omega} \tag{14}$$

Note that (12) defines a non standard cell-centered Finite-Volume scheme for computing both the unknown and the gradient from the knowledge of the laplacian.

3.3. *Second order error estimate*

Using the error estimates for the primal and dual schemes (7), (12), allows to derive a second order error estimate in the L^2 norm for the unknown u_h in (7). The proof uses an Aubin-Nitsche like argument.

Theorem 3 *The solution* (u_h, p_h) *of Scheme (7) verifies the optimal error estimate*

$$|u - u_h|_{0,\Omega} \leq Ch^2\,|f|_{0,\Omega} \tag{15}$$

3.4. *Further remarks*

A natural question is to ask whether there is the link between this scheme and the family of Finite Element mixed methods, [RT 77], [AB 85], [BDM 85], [BF 91]. In fact, it can be proved that the gradient part p_h in (7) coincides with the mixed gradient $\bar{p}_h$ in [RT 77]. However, the $\bar{u}_h$ part in the mixed method is only a P^0 approximation of u. The optimal error estimate is therefore only of first order in the L^2 norm. They are several methods in order to interpolate a posteriori $\bar{u}_h$ to an higher order approximation, [AB 85]. All these methods introduce a three variables problem $(\bar{u}_h, \bar{p}_h, \bar{\lambda}_h)$, involving a new degree of freedom (a Lagrange parameter) λ_h at the interfaces of the mesh. It can be proved, [Cr 99], that one of these interpolation coincides in fact with the scheme (7). Moreover, contrary to the standard mixed method in its original formulation, the degrees of freedom for u_h and p_h are decoupled, allowing to solve only an $O(NA)$ system in u_h, which is in addition symmetric definite positive. We refer to [CoC 98] for details onto the implementation of (7), and to [Cr 99] for the proofs of the results described here.

4. Conclusion

Several works are devoted to the a posteriori interpretation of mixed Finite Element methods as Finite Volume ones for the primal unknown $\bar{u}_h$. In other works, they are attempts to compute more easily this unknown, by introducing additional degrees of freedom. The advantage of the scheme (7) is that it is basically designed as a true Finite Volume scheme on a single computational cell, for u_h and p_h. In addition, it gives a natural decoupling between the unknowns u_h and p_h. Moreover, it has an optimal order of accuracy both for u_h and p_h, without any post-processing or a posteriori interpretation. Note that this kind of schemes is not restricted to triangular meshes, or to the dimension 2. We think that the formulation of this scheme as a Petrov-Galerkin method, combining the advantages of mixed and Finite Volume methods, can be particularly interesting for computations involving complex fluxes. Moreover, it can be of some help for a better understanding of the link between mixed and cell-centered Finite Volume methods. Higher order extensions are currently in progress.

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