Finite volume box schemes on triangular meshes

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Abstract : We introduce a finite volume box scheme for equation in divergence form $-div(\varphi(u)) = f$, which is a generalization of the box scheme of Keller. As in the Keller's scheme, affine approximations both of the unknow u and of the flux φ are used in each cell. Although the scheme is not variationnal, finite element spaces are used. We emphasize the case where the approximation spaces are the nonconforming P^1-space of Crouzeix-Raviart for the primary unknown u and the divergence conforming space of Raviart-Thomas for the flux φ. We prove an error estimate in the discrete energy seminorm for the Poisson problem. Finally, some numerical results and implementation details are given, proving that the scheme is effectively of second order.

Résumé : Nous introduisons un schéma boîte de type volume fini pour les équations sous forme divergence $-div(\varphi(u)) = f$, qui est une généralisation du schéma boîte de Keller. Comme dans le schéma de Keller, une approximation affine est utilisée dans chaque cellule, à la fois pour l'inconnue u et pour le flux φ. Bien que le schéma ne soit pas sous forme variationnelle, on utilise des espaces d'éléments finis. Nous décrivons plus particulièrement le cas où les espaces d'approximation sont l'espace P^1 non conforme de Crouzeix-Raviart pour l'inconnue primale et l'espace div-conforme de Raviart-Thomas pour le flux φ. Nous prouvons une estimation d'erreur en semi-norme d'énergie discrète pour le problème de Poisson. Finalement, la mise en œuvre de la méthode ainsi que quelques résultats numériques sont présentés, prouvant qu'elle est effectivement d'ordre 2.

Keywords : Box-method - Box-scheme - Finite volume scheme - Finite-element method - Mixed method - Raviart-Thomas element - Crouzeix-Raviart element - Poisson problem.
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1. Introduction

In a fundamental paper [17], H.B. Keller introduced the notion of box-scheme for parabolic equations. For an equation in divergence form, the main idea is to take the average of the conserved quantities on boxes defined from the mesh, in order to use only interface unknowns. The discretized equations form a so called *compact scheme*, in the sense that the local stencil of dependence of the scheme is reduced to the local "box".

The box-schemes of Keller have been applied by several authors [13,18] to non-standard parabolic equations, for example with moving boundaries, owning an integro-differential part, or involving constraints in some part of the domain. The results clearly demonstrate that the box-schemes are at least as good in precision than standard finite difference or finite element methods.

The box-schemes have been also used in some works in the 80' for compressible flows computations (Euler or Navier-Stokes equations). These schemes have indeed many interesting properties for the approximation of complex flows. They are conservative and of good accuracy for stationary solutions on relatively poor meshes. The matrices resulting from the discretization are compact and of simple structure on structured grids. Moreover there are no edge-gradient interpolation prolems as in the cell-centered finite-volume approach. We refer to Casier, Deconinck, Hirsch [6], Wornom [24,25], Wornom and Hafez [26], Chattot and Mallet[7], Courbet [9,10], Noye [22].

The aim of this paper is to introduce in a rigorous way a class of finite volume box-schemes on triangular meshes for equations in divergence form, like $\nabla.\varphi = f$, where the flux φ is given by a closure relation like $\varphi = F(u, \nabla u)$. The main interest of the new scheme is to allow an affine cell approximation both for the function u and for the flux φ, in the framework of a finite-volume method defined onto the primary mesh. This is clearly an important property when the closure model is complex. A typical example is when a large variation of the diffusion coefficients occurs whitin a cell, for example in boundary layers. The basic principles of the scheme are, firstly to remark that choosing the boxes as the primary triangular mesh gives the good number of equations [8], secondly to introduce a formulation mixing two types of standard finite element spaces : the nonconforming P^1 element of Crouzeix-Raviart [11] for the primary unknown and the divergence-conforming element of Raviart-Thomas of least order (RT_0) for the gradient [23]. The resulting scheme seems to be new. In particular, it is different from the classical mixed finite element approximation [23], which is variationnal, and insures the equality between unknowns and equations by a Babuska-Brezzi condition. It is also different from the box-scheme of Bank and Rose [1], also studied by Hackbusch [15]. This latter scheme remains basically variationnal and requires the construction of boxes as a dual mesh of the primary one. This is also the case in the covolume approach of Nicolaides [19,20,21]. Let us point out finally the recent works by Farhloul and Fortin

[14], and by Baranger, Maître, Oudin [2] on the connection between finite volume and mixed finite element methods. See also the work by Emonot, [12].

In the present paper, we restrict ourself to the presentation of the scheme onto the Poisson problem, i.e. when $\varphi = \nabla u$. The outline is as follows. After the introduction of the scheme in Section 2, we study in some details the particular case where the discrete spaces are the nonconforming P^1 space and the RT_0 space in Section 3. An error estimate in the energy semi-norm is derived. Finally we give in Section 4 some implementation details together with some numerical results, before to conclude in Section 5.

2. The principle of the scheme

Let us introduce the scheme on the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{onto } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain. The equation can be recasted in the mixed form with unknowns u and $\underline{p} = \nabla u$.

$$\begin{cases} \nabla \cdot \underline{p} + f = 0 & \text{in } \Omega \\ \underline{p} - \nabla u = 0 & \text{in } \Omega \\ u = 0 & \text{onto } \partial\Omega. \end{cases} \tag{2}$$

The problems (1) and (2) are equivalent and have a unique solution $(u, \underline{p}) \in (H_0^1(\Omega) \cap H^2(\Omega), H^1(\Omega)^2)$ when $f \in L^2(\Omega)$ and when Ω is convex or has a smooth boundary. Let $\mathcal{T}_h$ be a mesh consisting of triangles K, such that $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ with $\max\limits_{K \in \mathcal{T}_h} d(K)/\rho(K) \leq C$, where C is a constant independent of h, and $d(K), \rho(K)$ are the diameter of K and the diameter of the inscribed circle in K. We suppose that $d(K) \leq h$. We note $|K|$ the area of K, $A = A_i \cup A_b$ the set of the edges of $\mathcal{T}_h$ constitued of the internal edges A_i and the boundary edges A_b. The number of triangles is NE. The number of internal edges, boundary edges are NA_i, NA_b and the total number of edges is $NA = NA_i + NA_b$.

We approximate u by u_h and $\underline{p}$ by $\underline{p}_h$, where $u_h \in V_h$, and $\underline{p}_h \in Q_h$, V_h and Q_h being approximation spaces of finite element type. The consistency with (2) is not ensured in variationnal form but by the equations

$$\begin{cases} \text{(3a)} \ \left\langle \nabla \cdot \underline{p}_h + f, \mathbb{1}_K \right\rangle = 0 & \forall\ K \in \mathcal{T}_h \\ \text{(3b)} \ \left\langle \underline{p}_h - \nabla u_h, \mathbb{1}_K \right\rangle = 0 & \forall\ K \in \mathcal{T}_h \\ \text{(3c)} \ u_h = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

(3) is a finite volume method in that the trial functions $\mathbb{1}_K$ are indicatrices of the cells $K \in \mathcal{T}_h$. The equation (3a) can be rewritten as

$$\int_{\partial K} \underline{p}_h \cdot \underline{\nu} + |K| f_K = 0, \tag{4}$$

where $f_K = \frac{1}{|K|} \int_K f$ is the average of $f(x)$ on the triangle K. Thus, (3a) appears as a conservation law. Moreover the equation (3b) ensures in a weak sense the equality of ∇u_h and $\underline{p}_h$ in the triangle K.

3. The case $\mathbf{V_h}$= non conforming $\mathbf{P^1}$, $\mathbf{Q_h}$= $\mathbf{RT_0}$

3.1. *The approximation spaces*

We present in this section the standard approximation spaces of our scheme namely that where V_h is the non conforming $P1$ finite-element space of Crouzeix-Raviart, and Q_h the Raviart-Thomas space of least order (denoted RT_0). Recall that both spaces occur in classical finite element approximations of the Poisson equation, but not simultaneously. The non-conforming P^1 space is introduced in [11] for the Stokes problem, and can be used for the Poisson equation. No approximation of ∇u is required. On the other hand, the space RT_0 is introduced in [23] for the approximation of ∇u in the Poisson equation in mixed formulation, but the Babuska-Brezzi condition requires the P^0- approximation of u (i.e. constant in each triangle). For a good synthesis on these approximations, we refer to Braess [3], Brenner and Scott [4], Brezzi and Fortin [5].

Let us recall the definition of these two spaces. The space V_h is defined by

$$V_h = \{v_h / \forall\ K \in \mathcal{T}_h,\quad v_h|_K \in P_1(K),\quad v_h \text{ is continuous at the middle of each } e \in \partial K\}.$$

In other words, if $a \in \partial K_1 \cap \partial K_2$ is an edge of $\mathcal{T}_h$ and m_a the middle point of a, $v_h|_{K1}(m_a) = v_h|_{K2}(m_a)$. We denote by $(p_a(x))_{a\in A}$ the canonical basis of V_h, that is, the dual basis of the global degrees of freedom L_a defined by $\langle L_a, v_h\rangle = v_h(m_a)$. We have $\langle L_a, p_{a'}(x)\rangle = \delta_{aa'}$ for $a, a' \in A$. If $u_h(x) = \sum_{a\in A} u_a p_a(x)$, the restriction of u_h to the triangle K is given by

$$u_h(x)/_K = \sum_{e\in\partial K} u_e p_e(x),$$

where $p_e(x) = 1 - 2\lambda_S(x)$, $\lambda_S(x)$ being the barycentric coordinate of x with respect to S, vertex opposite to e in the triangle K. Note that $\nabla p_e(x) = \frac{|e|}{|K|}\underline{\nu}_e$.

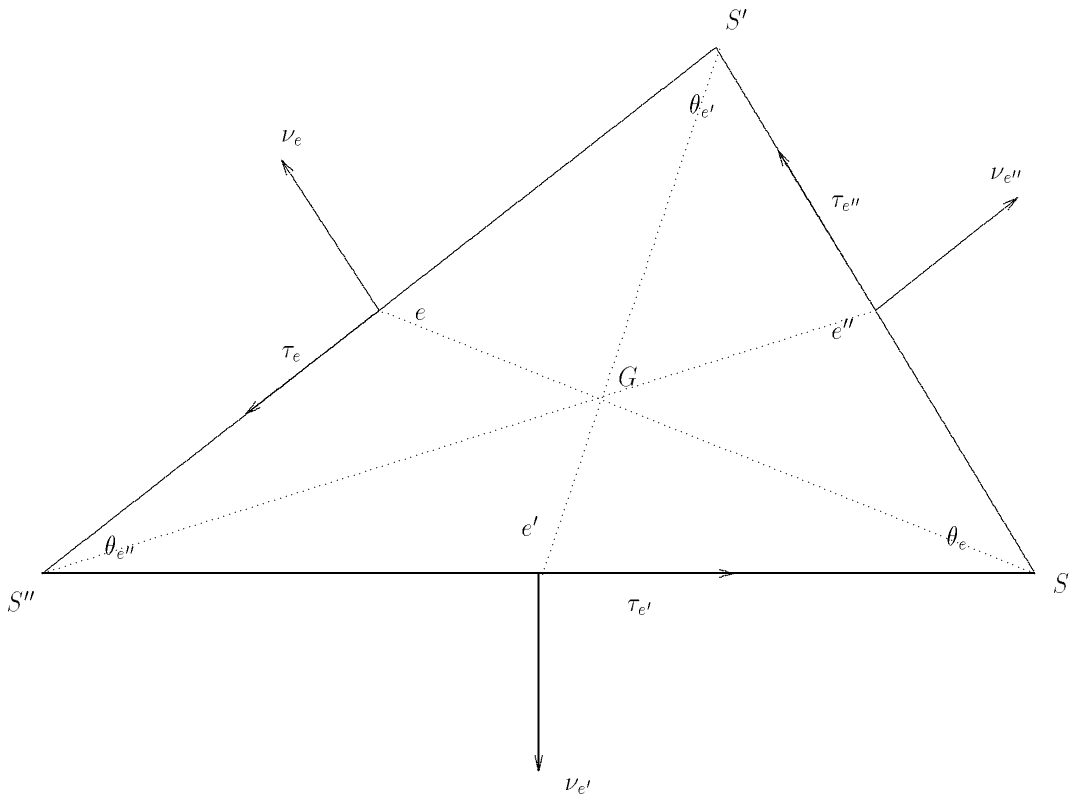

Fig. 1. *A triangle of* $\mathcal{T}_h$

Moreover, we denote by $V_{h,0}$ the subspace of the $u_h \in V_h$ such that $u_a = 0$ for each edge $a \in A_b$.

The space Q_h is defined by

$$Q_h = \{\underline{q}_h(x) \in H_{\mathrm{div}}(\Omega) \; / \; \forall \; K \in \mathcal{T}_h, \; \underline{q}_h(x)|_K \in RT_0(K)\}$$

where, for each $K \in \mathcal{T}_h$, $RT_0(K) = P_0(K)^2 + P_0(K) \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$ (dim $RT_0(K) = 3$). The constraint $\underline{q}_h(x) \in H_{\mathrm{div}}(\Omega)$ is equivalent to the continuity of the normal component $\underline{q}_h \cdot \underline{\nu}_a$ through each edge $a = K_1 \cap K_2$. If $a = e$ in K_1 and $a = e'$ in K_2, one has

(5) $$\underline{q}_h|_{K_1}(x) \cdot \underline{\nu}_e + \underline{q}_h|_{K_2}(x) \cdot \underline{\nu}_{e'} = 0, \quad \forall \; x \in a.$$

The global degrees of freedom of Q_h are the linear forms L_a, $a \in A$, defined by

$$\left\langle L_a, \underline{q}_h \right\rangle = \int_a \underline{q}_h \cdot \underline{\nu}_a d\sigma \quad \text{(circulation of } \underline{q}_h \text{ along the edge } a).$$

The canonical basis of Q_h (dual basis of $(L_a)_{a\in A}$) is given by

$$\underline{P}_a(x) = \underline{P}_{K_1,e}(x)\mathbb{1}_{K_1}(x) - \underline{P}_{K_2,e'}(x)\mathbb{1}_{K_2}(x),$$

where a is oriented from K_1 towards K_2, $a = e$ in K_1, $a = e'$ in K_2. For each $K \in \mathcal{T}_h$, and each $e \in \partial K$, the polynome $\underline{P}_{K,e}$ is defined by

$$\underline{P}_{K,e}(x) = \frac{1}{2|K|}\begin{bmatrix} x^1 - x_S^1 \\ x^2 - x_S^2 \end{bmatrix}, \ \forall\, x = (x^1, x^2) \in K.$$

Note that, for $x \in a$, $\underline{P}_{K,e}(x) \cdot \underline{\nu}_e = \frac{1}{|a|}$.

Moreover, if $\underline{q}_h \in Q_h$ is globally decomposed onto $(\underline{P}_a)_{a\in A}$ in the form

$$\underline{q}_h(x) = \sum_{a\in A} q_a \underline{P}_a(x),$$

then the local decomposition of $\underline{q}_h(x)|_K$ onto $(\underline{P}_{K,e})_{e\in\partial K}$ is

$$\underline{q}_h(x)|_K = \sum_{e\in\partial K} q_e \ \underline{P}_{K,e}(x),$$

where $q_e = q_{a(K,e)}$ if the global orientation of e is from $K = K_1$ towards K_2, and $q_e = -q_{a(K,e)}$ in the opposite case.

Finally, $\underline{q}_h(x)|_K$ admits also a useful representation in the form ([2])

$$\underline{q}_h(x)|_K = q_K + |K|(\nabla \cdot \underline{q}_h)_K \underline{P}_K(x) \tag{6}$$

where $q_K = \frac{1}{|K|}\int_K \underline{q}_h$, $(\nabla \cdot \underline{q}_h)_K$ is the constant value of $\nabla \cdot \underline{q}_h$ in K, and $\underline{P}_K(x)$ is the polynome of first order

$$\underline{P}_K(x) = \frac{1}{3}\sum_{e\in\partial K} \underline{P}_e(x) = \frac{1}{2|K|}\begin{bmatrix} x^1 - x_G^1 \\ x^2 - x_G^2 \end{bmatrix}, \ \forall\, x \in K.$$

3.2. *The discrete system*

Let us describe now the discrete Poisson equation obtained in the case where V_h is the non conforming P^1space and Q_h is the RT_0-space. Let $u_h \in V_h$ and $\underline{p}_h \in Q_h$ have the local decomposition on each $K \in \mathcal{T}_h$,

$$u_h(x) = \sum_{e\in\partial K} u_e p_e(x), \quad \underline{p}_h(x) = \sum_{e\in\partial K} p_e \underline{P}_e(x).$$

Equation (3a) gives for $K \in \mathcal{T}_h$

$$(7a) \qquad 0 = \int_{\partial K} \underline{p}_h \cdot \underline{\nu} + |K| f_K = \sum_{e \in \partial K} p_e + |K| f_K \quad (NE \text{ equations})$$

Equation (3b) gives

$$0 = \int_K (\underline{p}_h - \nabla u_h) = \sum_{e \in \partial K} p_e \int_K \underline{P}_e(x) - u_e \int_K \nabla p_e(x).$$

Recalling that $\nabla p_e(x) = \frac{|e|}{|K|} \underline{\nu}_e$ and denoting $\underline{Q}_e = \int_K \underline{P}_e(x)$, $\underline{N}_e = |e| \underline{\nu}_e$ we get, for each $K \in \mathcal{T}_h$,

$$(7b) \qquad 0 = \sum_{e \in \partial K} \left[p_e \underline{Q}_e - u_e \underline{N}_e \right] \quad (2\ NE \text{ equations}).$$

Note that since

$$\sum_{e \in \partial K} \underline{Q}_e = \int_K \sum_{e \in \partial K} \underline{P}_e(x) = 3 \int_K \underline{P}_K(x) = 0,$$

we have $\underline{Q}_{e_3} = -(\underline{Q}_{e_1} + \underline{Q}_{e_2})$. Moreover we have $\sum_{e \in \partial K} \underline{N}_e = 0$. Finally the Dirichlet boundary condition gives, for each $a \in \partial \Omega$

$$(7c) \qquad 0 = u_a.$$

More generally, we will consider boundary conditions of the form, for $a \in A_b$,

$$0 = \langle B_{a,u}, u_h \rangle + \left\langle B_{a,p}, \underline{p}_h \right\rangle \qquad (NA_b \text{ equations}),$$

where $B_{a,u}, B_{a,p}$ are linear forms onto V_h, Q_h such that at least one of $B_{a,u}, B_{a,p}$ is different from 0. For example, a mixed boundary condition on the edge $a \in A_b$ will give

$$(7d) \qquad m_a u_a + \ell_a p_a = n_a$$

where $(m_a, \ell_a) \neq (0,0)$. A Neumann boundary condition is given by $m_a = 0$, $\ell_a = 1$. By counting the edges of $\mathcal{T}_h$ we have

$$3NE = \sum_K \sum_{e \in \partial K} 1 = 2 \sum_{a \in A_i} 1 + \sum_{a \in A_b} 1 = 2NA - NA_b.$$

Thus, we get the relation between the number of triangles NE, the total number of edges NA, and the number of boundary edges NA_b

$$(8) \qquad 3NE + NA_b = 2NA.$$

The number of unknowns $(u_a, p_a)_{a\in A}$ is equal to the number of the equations (7a),(7b),(7c).

We note finally that the relation (6) gives the following representation of $\underline{p}_h(x)$ in each triangle K

$$\underline{p}_h(x) = \nabla u_K - |K| f_K \underline{P}_K(x), \tag{9}$$

where we note $\nabla u_K = \frac{1}{|K|}\int_K \nabla u_h$.

Summarizing the discrete system (7a,b,c), we get the discrete problem : Find $u_h(x) = \sum_{a\in A} u_a p_a(x)$, $\underline{p}_h(x) = \sum_{a\in A_b} p_a \underline{P}_a(x)$ such that

$$\begin{cases} \displaystyle\sum_{e\in\partial K} p_e + |K| f_K = 0 & \forall\ K \in \mathcal{T}_h \\ \displaystyle\sum_{e\in\partial K} \left[p_e \underline{Q}_e - u_e \underline{N}_e\right] = 0 & \forall\ K \in \mathcal{T}_h \\ u_a = 0 & \forall\ a \in A_b. \end{cases} \tag{10}$$

Note finally the following elementary result, linking the 3 vectors $(\underline{Q}_e)_{e\in\partial K}$ and $(\underline{N}_e)_{e\in\partial K}$ (see Fig.1 for the notations)

$$\underline{Q}_e = \frac{1}{3}(\text{cotan } \theta_e \underline{N}_e - \frac{1}{2}\text{cotan } \theta_{e'} \underline{N}_{e'} - \frac{1}{2}\text{cotan } \theta_{e''} \underline{N}_{e''})$$

3.3. *Numerical analysis*

This section is devoted to the numerical analysis of the problem (1) approximated by the discrete system (10). The main tools are those of the finite element method, although the framework is not of variational type.

Let us introduce some standard notations.

$$\begin{aligned} |u|_{0,\Omega} &= \left[\int u^2(x)dx\right]^{1/2} \quad \text{for } u \in L^2(\Omega) \\ |u|_{m,\Omega} &= \left[\int |D^m u(x)|^2\, dx\right]^{1/2} \quad \text{for } u \in H^m(\Omega) \\ \|u\|_{h,\Omega} &= \left(\sum_K \int_K |\nabla u|^2 dx\right)^{1/2} \quad \text{for } u \in\ H^1(\Omega) \oplus V_h. \end{aligned}$$

The first observation is

Lemma 1 [1]. *The discrete energy semi-norm* $\|v_h\|_h$ *is a norm onto the space* $V_{h,0} = \{v_h \in V_h,\, v_h = 0 \text{ on } \partial\Omega\}$.

Proof : Let $v_h \in V_{h,0}$ such that $\|v_h\|_h = 0$. The gradient of v_h is zero in each cell $K \in \mathcal{T}_h$. Hence v_h is constant in each K. Since v_h is continuous at the middle of each edge a of $\mathcal{T}_h$ and $v_h = 0$ onto $\partial\Omega$, we deduce that $v_h = 0$ in Ω. ■

The first result is the existence and unicity of the discrete problem (10).

Theorem 1. *The discrete problem* (10) *has a unique solution* $(u_h, \underline{p}_h) \in V_{h,0} \times Q_h$.

Proof : The problem (10) in $(u_h, \underline{p}_h) \in V_{h,0} \times Q_h$ is linear, and the number of unknowns is equal to the number of equations. Hence, it is sufficient to prove that $f = 0$ implies $u_h = \underline{p}_h = 0$. The relation (9) gives that $\underline{p}_h(x)$ is a constant $\underline{c}_K$ in each $K \in \mathcal{T}_h$ and that $\underline{c}_K = \nabla u_K$. Hence

$$\begin{aligned}
\left|\underline{p}_h\right|_{0,\Omega}^2 &= \sum_K |K|\, |c_K|^2 = \sum_K |K| c_K \cdot \nabla u_K \\
&= \sum_K \int_K \underline{p}_h(x) \cdot \nabla u_h(x) dx \\
&= \sum_K \int_{\partial K} (\underline{p}_h(x) \cdot \underline{\nu}(x)) u_h(x) d\sigma - \int_K \nabla \cdot \underline{p}_h(x) u_h(x) dx.
\end{aligned}$$

Since $\nabla \cdot \underline{p}_h(x)|_K = f_K = 0$, and $u_h \equiv 0$ on $\partial\Omega$,

$$\begin{aligned}
\left|\underline{p}_h\right|_{0,\Omega}^2 &= \sum_K \int_{\partial K} (\underline{p}_h(x) \cdot \underline{\nu}(x)) u_h(x) d\sigma \\
&= \sum_{a \in A_i} \int_a (\underline{p}_{h,1} \cdot \underline{\nu}_a) u_{h,1} - (\underline{p}_{h,2} \cdot \underline{\nu}_a) u_{h,2},
\end{aligned}$$

where A_i is the set of the internal edges and the edge a is oriented from K_1 towards K_2. Denoting by p_a the constant value of $\underline{p}_{h,1}(x) \cdot \underline{\nu}_a = \underline{p}_{h,2}(x) \cdot \underline{\nu}_a$ for $x \in a$, one has

$$\left|\underline{p}_h\right|_{0,\Omega}^2 = \sum_{a \in A_i} p_a \int_a (u_{h,1} - u_{h,2}) = 0$$

by definition of V_h. Therefore $\underline{c}_K = \nabla u_K = 0$ for each K, hence $\|u_h\|_h = 0$ and by Lemma 1, $u_h = 0$. ■

Before proving an error estimate, note the two following stability estimates :

Proposition 1. *If* $(u_h, \underline{p}_h) \in V_{h,0} \times Q_h$ *is the solution of* (10), *then*

$$\text{(11)} \qquad \text{(i)} \ \|u_h\|_h \le \left|\underline{p}_h\right|_{0,\Omega} \le C\left(\|u_h\|_h + h|f|_{0,\Omega}\right), \ C \text{ independent of } h,$$

$$\text{(12)} \qquad \text{(ii)} \ \left\|\underline{p}_h\right\|_h \le \frac{1}{2^{1/2}}|f|_{0,\Omega}.$$

<u>Proof</u> : (i) The equality (3b) gives $\nabla u_K = \frac{1}{|K|}\int_K \underline{p}_h(x)dx$, hence

$$\|u_h\|_h^2 = \sum_K |K|\,|\nabla u_K|^2 \le \sum_K \int_K \left|\underline{p}_h(x)\right|^2 dx = \left|\underline{p}_h\right|_{0,\Omega}^2.$$

Moreover, (9) gives

$$\left|\underline{p}_h\right|_{0,K} \le \|u_h\|_{h,K} + |K|\,|f_K|\,|\underline{P}_K|_{0,K}\,.$$

We have

$$|\underline{P}_K|_{0,K}^2 = \frac{1}{4|K|^2}\int_K (x^1 - x_G^1)^2 + (x^2 - x_G^2)^2 = \frac{\rho_K^2}{4|K|}$$

where ρ_K is the gyration radius of K. By noting that the regularity assumption on the mesh insures the existence of $\bar{C}$, independent of h, such that $\sup\limits_K \frac{\rho_K}{|K|^{1/2}} \le \bar{C}$ and that $|f_K| \le \frac{1}{|K|^{1/2}}|f|_{0,K}$, we get by summation on $K \in \mathcal{T}_h$,

$$\left|\underline{p}_h\right|_{0,\Omega} \le C\left(\|u_h\|_h + h|f|_{0,\Omega}\right)$$

where $C = \max(2^{1/2}, \bar{C}/2^{1/2})$.
(ii) Again (9) gives

$$\nabla\underline{p}_h(x)|_K = |K| f_K \nabla\underline{P}_K.$$

Thus

$$\left\|\underline{p}_h\right\|_h^2 = \sum_K \left|\nabla\underline{p}_h\right|_{0,K}^2 = \sum_K |K|^2\,|f_K|^2\,|\nabla\underline{P}_K|_{0,K}^2$$

Noting that $|\nabla\underline{P}_K|_{0,K}^2 = \frac{1}{2|K|}$, we obtain

$$\left\|\underline{p}_h\right\|_h^2 = \frac{1}{2}\sum_K |K|\,|f_K|^2 \le \frac{1}{2}|f|_{0,\Omega}^2. \quad \blacksquare$$

Our second main result is an error estimate in the discrete energy norm $\|\ \|_h$. Let $u \in H^2 \cap H_0^1$ be the solution of the Poisson problem (1) with $f \in L^2(\Omega)$. We consider also $\underline{p}(x) \in H^1(\Omega)^2$ defined by $\underline{p}(x) = \nabla u(x)$. For $u, v \in H^1 \oplus V_h$ we define

$$a(u,v) = \sum_K \int_K \nabla u \cdot \nabla v$$

the bilinear form associated with $\| \ \|_{h,\Omega}$. On $H(\mathrm{div},\Omega) = \{\underline{p} \in L^2(\Omega)^2 / \nabla \cdot \underline{p} \in L^2(\Omega)\}$ we define the semi-norm

$$|\underline{p}|^2_{\mathrm{div},\Omega} = \int_\Omega (\nabla \cdot \underline{p})^2 dx$$

associated with the bilinear form

$$b(\underline{p}, \underline{q}) = \int_\Omega (\nabla \cdot \underline{p})(\nabla \cdot \underline{q}) dx.$$

Theorem 2. *There exist constants $C = C(\Omega) > 0$ independent of h such that*

(i) $$\|u - u_h\|_h \leq Ch|u|_{2,\Omega}$$

(ii) $$\left|\underline{p} - \underline{p}_h\right|_{0,\Omega} \leq Ch|u|_{2,\Omega}$$

(iii) $$\left|\underline{p} - \underline{p}_h\right|_{\mathrm{div},\Omega} \leq Ch|u|_{3,\Omega}.$$

Proof of (i) : We follow a classical strategy. We have for any $v_h \in V_{h,0}$

$$\begin{aligned} \|u - u_h\|_h &\leq \|u - v_h\|_h + \|u_h - v_h\|_h \\ \|u_h - v_h\|_h^2 &= a(u_h - v_h, u_h - v_h) \\ &= a(u_h - u, u_h - v_h) + a(u - v_h, u_h - v_h). \end{aligned} \tag{13}$$

Thus

$$\|u_h - v_h\|_h \leq \sup_{v_h \in V_{h,0}} \frac{|a(u_h - u, u_h - v_h)|}{\|u_h - v_h\|_h} + \|u - v_h\|_h$$

and (13) gives

$$\|u - u_h\|_h \leq 2 \inf_{v_h \in V_{h,0}} \|u - v_h\|_h + \sup_{w_h \in V_{h,0}} \frac{|a(u_h - u, w_h)|}{\|w_h\|_h}. \tag{14}$$

Since the space $V_{h,0}$ contains the standard P^1-Lagrange finite element space, the classical interpolation estimates gives $\inf_{v_h \in V_{h,0}} \|u - v_h\|_h \leq C(\Omega) h |u|_{2,\Omega}$. It remains to estimate the second term. We have

$$a_h(u_h - u, w_h) = \sum_K \left[\int_K \nabla u_h \cdot \nabla w_h - \int_K \nabla u \cdot \nabla w_h \right]. \tag{15}$$

∇u_h is constant on each K, and by (3b) its value is $\underline{p}_{h,K} = \frac{1}{|K|} \int_K \underline{p}_h(x) dx$. Thus

$$\begin{aligned} \int_K \nabla u_h \cdot \nabla w_h &= \int_K \underline{p}_h(x) \cdot \nabla w_h(x) dx \\ &= - \int_K \nabla \cdot \underline{p}_h(x) w_h(x) + \int_{\partial K} w_h(x) \underline{p}_h(x) \cdot \underline{\nu}(x) d\sigma. \end{aligned}$$

(3a) gives $\int_K \nabla \cdot \underline{p}_h(x) + f(x) = 0$. Thus the value of the constant $\nabla \cdot \underline{p}_h(x)$ in K is $-f_K$ where $f_K = \frac{1}{|K|}\int_K f$. Hence

$$\int_K \nabla u_h \cdot \nabla w_h = \int_K f_K w_h(x) + \int_{\partial K} w_h(x)\underline{p}_h(x) \cdot \underline{\nu}(x)\ d\sigma.$$

Moreover

$$\begin{aligned}\int_K \nabla u \cdot \nabla w_h &= \int_K -\Delta u\ w_h + \int_{\partial K} \frac{\partial u}{\partial \nu} w_h \\ &= \int_K f\ w_h + \int_{\partial K} \frac{\partial u}{\partial \nu} w_h.\end{aligned}$$

Thus (15) can be rewritten as

$$\sum_K \int_K \underset{\text{(I)}}{[f_K - f(x)]\, w_h(x) dx} + \sum_K \int_{\partial K} \underset{\text{(II)}}{\left[\underline{p}_h(x) - \nabla u(x)\right] \cdot \underline{\nu}\ w^h(x) d\sigma(x)} \tag{16}$$

Since $\int_K f_K - f(x) = 0$, one can substract a constant value from $w_h(x)$ in each term of the first sum and rewrite (I) as

$$\text{(I)} = \sum_K \int_K (f_K - f(x))(w_h(x) - w_{h,K}) dx.$$

Thus

$$\begin{aligned}|\text{(I)}| &\leq \sum_K |f_K - f|_{0,K}\, |w_h - w_{h,K}|_{0,K} \\ &\leq Ch|f|_{0,\Omega}\, \|w_h\|_h \\ &\leq Ch|u|_{2,\Omega}\, \|w_h\|_h\,.\end{aligned}$$

Consider now the sum (II) in (16). Each internal edge $e \in \partial K$ occurs two times in the sum with a vector $\underline{\nu}$ changing of sign. On each boundary edge e, one has $\int_e w_h d\sigma = 0$ since $w_h \in V_{h,0}$. Thus, by substracting the function $(\frac{1}{|e|}\int_e (\underline{p}_h(x) - \nabla u(x)) \cdot \underline{\nu}_e\ d\sigma) w_h(x)$, we do not change the sum. Its value is

$$\begin{aligned}&\sum_K \int_{\partial K} \left[\underline{p}_h(x) - \nabla u(x)\right] \cdot \underline{\nu}\ w_h(x) d\sigma = \\ &\sum_K \sum_{e \in \partial K} \int_e \left[(\underline{p}_h(x) - \nabla u(x)) \cdot \underline{\nu}_e - \frac{1}{|e|}\int_e (\underline{p}_h(x) - \nabla u(x)) \cdot \underline{\nu}_e\right] w_h(x) d\sigma.\end{aligned} \tag{17}$$

Recall now the following result (Lemma 3 of [11]).

Lemma 2. *Let $e \in \partial K$, $v, \varphi \in H^1(K)$, $v_e = \frac{1}{|e|}\int_e v(x) d\sigma$, then*

$$\left|\int_e \varphi(v - v_e) d\sigma\right| \leq Ch|\varphi|_{1,K}|v|_{1,K},$$

where C is independent of h.

Applying this result to the right-hand side of (17) gives

$$|(\mathrm{II})| \le Ch \sum_K \left|\underline{p}_h - \nabla u\right|_{1,K} |w_h|_{1,K} \le Ch \left\|\underline{p}_h - \nabla u\right\|_h \|w_h\|_h ,$$

and, using (12)

$$|(\mathrm{II})| \le Ch\left[|f|_{0,\Omega} + |u|_{2,\Omega}\right] \|w_h\|_h \le 2Ch|u|_{2,\Omega} \|w_h\|_h .$$

Finally, there exists $C > 0$ independent of h such that

$$\sup_{w_h \in V_{h,0}} \frac{|a(u_h - u, w_h)|}{\|w_h\|_h} \le |(\mathrm{I})| + |(\mathrm{II})| \le Ch|u|_{2,\Omega}.$$

Going back to (14), we obtain

$$\|u - u_h\|_h \le Ch|u|_{2,\Omega}.$$

<u>Proof of (ii)</u> : From the representation identity (9) of $\underline{p}_h(x)|_K$ we have

$$\underline{p}_h(x)|_K = \nabla u_{h,K} - |K| f_K \underline{P}_K(x) \quad \text{and} \quad \underline{p}(x) = \nabla u(x).$$

Thus

$$\underline{p}_h(x)|_K - \underline{p}(x)|_K = \nabla u_{h,K} - \nabla u(x) - |K| f_K \underline{P}_K(x)$$

and

$$\left|\underline{p}_h - p\right|_{0,K} \le |\nabla u_h - \nabla u|_{0,K} + |K|\, |f_K|\, |\underline{P}_K|_{0,K} .$$

Since $|\underline{P}_K|_{0,K} = \frac{\rho_K}{2|K|^{1/2}} \le \frac{\bar{C}}{2}$and $|f_K| \le \frac{1}{|K|^{1/2}} |f|_{0,K}$, we deduce

(18) $$\left|\underline{p}_h - \underline{p}\right|_{0,\Omega} \le \|u_h - u\|_h + Ch|f|_{0,\Omega} \le Ch|u|_{2,\Omega},$$

where C stands for a constant independent of h.

<u>Proof of (iii)</u> : Again by (9), $\nabla \cdot \underline{p}_h(x)|_K = -|K| f_K \nabla \cdot \underline{P}_K(x) = -f_K$ and $\nabla \cdot \underline{p} = -f(x)$. Thus, $\left|\nabla \cdot \underline{p}_h - \nabla \cdot \underline{p}\right|_{0,K} = |f - f_K|_{0,K} \le Ch|f|_{1,K}$ and, by summation over the $K \in \mathcal{T}_h$, we obtain

(19) $$\left|\nabla \cdot \underline{p}_h - \nabla \cdot \underline{p}\right|_{0,\Omega} \le Ch|f|_{1,\Omega} \le Ch|u|_{3,\Omega}. \qquad \blacksquare$$

Since $V_{h,0} \not\subset H_0^1$ we can't deduce directly from Theorem 2(i) an error estimate in the L^2 norm by the Poincaré inequality. We propose a regularity assumption on the triangulation $\mathcal{T}_h$, which is sufficient to insure such an inequality.

Hypothesis (H)*:* There exists a disjoint cover of $\mathcal{T}_h$ by a set of N_h connected slabs $\mathcal{B}_i$ where each slab $\mathcal{B}_i$ is made of $N_{i,h}$ triangles, with at least one triangle in contact with the boundary $\partial\Omega$. Moreover

$$\text{(H1)} \qquad N_h = O\left(\frac{1}{h}\right),$$

$$\text{(H2)} \qquad \sup N_{i,h} = O\left(\frac{1}{h}\right).$$

This hypothesis can be read as a type of structuration of $\mathcal{T}_h$. The triangulation of Figure 2 satisfies this hypothesis.

Lemma 3 : *Under the hypothesis* (H) *on the triangulation* $\mathcal{T}_h$*, there exists* $C(\Omega) > 0$ *such that for* $u \in H_0^1 \oplus V_{h,0}$

$$|u|_{0,\Omega} \leq C(\Omega)\|u\|_h.$$

<u>Proof</u> : Since this inequality is true for $u \in H_0^1$ (Poincaré inequality), it is sufficient to prove it for $u \in V_{h,0}$. Let $u \in V_{h,0}$. For each $x \in \mathcal{B}_i$, consider the path $\gamma \subset \mathcal{B}_i$, γ being defined by $[x_0, x_1] \cup [x_1, x_2] \cup ... \left[x_{N_i(x)}, x\right]$ where the x_j are mid-edge points of the triangles of $\mathcal{B}_i$ and where $x_0 \in \partial\Omega \cap \mathcal{B}_i$.

By definition of $V_{h,0}$, u_h/γ is piecewise affine and continuous ; hence

$$\begin{aligned}|u(x)| &\leq \sum_{j=1}^{N_{i(x)}-1} \left|\nabla u_{K_j}\right| |x_j - x_{j-1}| + \left|\nabla u_{N_i(x)}\right| \left|x - x_{N_i(x)}\right| \\ &\leq Ch \sum_{j=1}^{N_{i,h}} \left|\nabla u_{K_j}\right|.\end{aligned}$$

Taking the L^2 norm of u on $\mathcal{B}_i$, gives

$$|u|_{0,\mathcal{B}_i} \leq Ch \left|\mathcal{B}_i\right|^{1/2} \sum_{j=1}^{N_{i,h}} \left|\nabla u_{K_j}\right|.$$

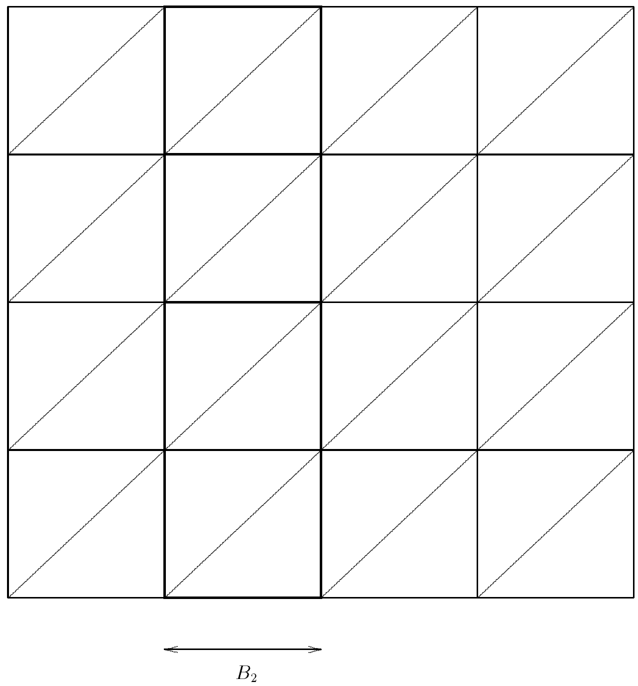

Fig. 2. *A triangulation $\mathcal{T}_h$ satisfying the hypothesis (H) with* $N_h = \frac{1}{h}$; $N_{i,h} = \frac{2}{h}$; $M = 1$.

Since $N_h = O\left(\frac{1}{h}\right)$ by (H1), we have $|\mathcal{B}_i| = O(h)$ and the Cauchy-Schwarz inequality yields

$$|u|_{0,\mathcal{B}_i} \leq Ch^{1/2}\left(\sum_{j=1}^{N_{i,h}} h^2 \left|\nabla u_{K_j}\right|^2\right)^{1/2} N_{i,h}^{1/2}.$$

Moreover $N_{i,h} = O\left(\frac{1}{h}\right)$ (hypothesis H2), hence

$$|u|_{0,\mathcal{B}_i} \leq C\|u\|_{h,\mathcal{B}_i}.$$

Summation over the $\mathcal{B}_i$ yields the conclusion since the $\mathcal{B}_i$ are a disjoint cover of $\mathcal{T}_h$.
■

Theorem 2(i) and Lemma 2 allow the L^2 error estimate

Corollary 1. *Under the hypothesis* (H) *on the mesh* $\mathcal{T}_h$, *there exists* C *independent of* h *such that*

$$|u - u_h|_{0,\Omega} \leq Ch|u|_{2,\Omega}.$$

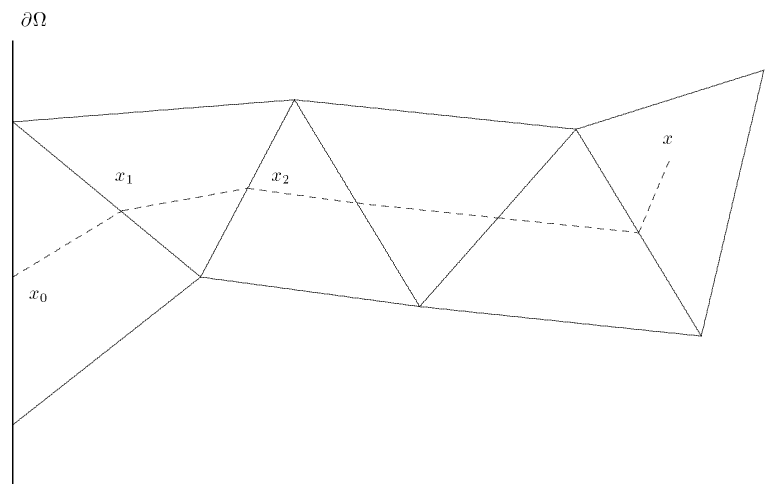

Fig. 3. *A path joining* $x \in K$ *to* $\partial\Omega$

4. Numerical results

4.1. Implementation

We present in this section the principle of the implementation of the discrete system (10). We call $U = (u_a)_{a\in A}$ the vector of the components of $u_h(x)$ onto the P^1 non-conforming global basis $p_a(x)$ (see §3.1). We define also U_K and P_K the vectors of the local components in the cell K of $u_h(x)$ and $\underline{p}_h(x)$.

$$U_K = [u_{e_1}, u_{e_2}, u_{e_3}]^T, \quad P_K = [p_{e_1}, p_{e_2}, p_{e_3}]^T,$$

where $\partial K = \{e_1, e_2, e_3\}$ are the 3 edges of K. (No specific orientation of the 3 edges is required in U_K and P_K). Clearly (10) can be rewritten as

$$-\tilde{L}_K \cdot U_K + \tilde{M}_K \cdot P_K = -\tilde{N}_K \tag{24}$$

where $\tilde{L}_K, \tilde{M}_K \in M_3(\mathbb{R}),\ \tilde{N}_K \in \mathbb{R}^3$ are

$$\tilde{L}_K = \frac{1}{|K|}\begin{bmatrix} 0 & 0 & 0 \\ \underline{N}^x_{e_1} & \underline{N}^x_{e_2} & \underline{N}^x_{e_3} \\ \underline{N}^y_{e_1} & \underline{N}^y_{e_2} & \underline{N}^y_{e_3} \end{bmatrix} \quad \tilde{M}_K = \frac{1}{|K|}\begin{bmatrix} 1 & 1 & 1 \\ Q^x_{e_1} & Q^x_{e_2} & Q^x_{e_3} \\ Q^y_{e_1} & Q^y_{e_2} & Q^y_{e_3} \end{bmatrix}$$

$$\tilde{N}_K = \begin{bmatrix} f_K \\ 0 \\ 0 \end{bmatrix}.$$

Since $\underline{Q}_{e_3} = -(\underline{Q}_{e_1} + \underline{Q}_{e_2})$, we deduce that the 3 vectors of $\mathbb{R}^3$ $(1, \underline{Q}_{e_1}), (1, \underline{Q}_{e_2}), (1, \underline{Q}_{e_3})$ are never colinear. Hence $\tilde{M}_K$ is non singular and (24) can be rewritten as

$$P_K = -N_K + L_K \cdot U_K \tag{25}$$

where $N_K = \tilde{M}_K^{-1}\tilde{N}_K,\ L_K = \tilde{M}_K^{-1}\tilde{N}_K$.

We eliminate now the unknowns $(p_a)_{a\in A}$. If a is an internal edge, with orientation from $K_1(a)$ towards $K_2(a)$, $a = e_1$ in $K_1(a)$, $a = e_2$ in $K_2(a)$, the identity $P_{K_1,e_1} = -P_{K_2,e_2}$ holds. Thus we have

$$[L_{K_1} \cdot U_{K_1}]_{e_1} + [L_{K_2} \cdot U_{K_2}]_{e_2} = N_{K_1,e_1} + N_{K_2,e_2} \tag{25_1}$$

Consider now a boundary edge $a \in \partial K_1$ with boundary condition (7d)

$$m_a u_a + \ell_a p_a = n_a$$

there are two cases, corresponding respectively to Neumann and Dirichlet boundary conditions :

(i) $\qquad \ell_a \neq 0,\ $ then $p_a = \dfrac{1}{\ell_a}(n_a - m_a u_a) = [-N_{K_1} + L_{K_1} \cdot U_{K_1}]_a$

(ii) $\qquad \ell_a = 0,\ $ then $m_a \neq 0$ and $u_a = \dfrac{n_a}{m_a}$

We obtain in this way a linear system in the unknown $U = (u_a)_{a\in A}$

$$\mathcal{A}U = b \tag{26}$$

where $\mathcal{A}$ is the global stiffness matrix and b the global right hand side.

The final algorithm is similar to the one of the standard finite element method, with a main loop on the elements. It can be written shortly

do for $K \in \mathcal{T}_h$

evaluate L_K, N_K

assemble the contribution of L_K *to* $\mathcal{A}$, N_K *to* B

enddo

do resolution of $\mathcal{A}U = b$.

If necessary, $\underline{p}_h(x)$ can be evaluated from $u_h(x)$ by (25).

We define now $U_i \in I\!R^{NA_i}$ the subvector of $U \in I\!R^{NA}$ corresponding to the internal degrees of freedom (i.e. the internal edges). $\mathcal{A}_i$ is the matrix extracted from $\mathcal{A}$ that has the same dimension that U_i and $b_i \in I\!R^{NA_i}$ is the corresponding right hand side. In the case of the homogeneous Dirichlet problem, the resolution of $\mathcal{A}U = b$ is equivalent to the system $\mathcal{A}_i U_i = b_i$. It is not directly apparent from the form of the elementary matrices $\tilde{L}_K, \tilde{M}_K$ that the matrix $\mathcal{A}_i$ is symmetric definite positive.

Proposition 2 *The global siffness matrix* $\mathcal{A}_i$ *corrsponding to the internal degrees of freedom of the system (26) is symmetric positive definite*

Proof : For each $K \in \mathcal{T}_h$, an easy calculation shows that the 3x3 matrix L_K and that the vector N_K are

$$L_K = 2\begin{bmatrix} c_2+c_3 & -c_3 & -c_2 \\ -c_3 & c_3+c_1 & -c_1 \\ -c_2 & -c_1 & c_1+c_2 \end{bmatrix} \quad ; \quad N_K = \frac{|K|}{3} f_K \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $c_i = \text{cotan}\ \theta_{e_i}, i = 1,2,3$. This can be checked either directly from (24), or by integrating the relation (9) along each edge $e \in \partial K$. Using the fact that $c_2 + c_3 \geq 0$, $c_1c_2 + c_2c_3 + c_3c_1 = 1$, we deduce that the 2 first minors of L_K are non-negative, hence L_K is a rank 2 symmetric positive matrix.

We introduce now $\hat{L}_K$ the $NA_i \times NA_i$ matrix, and $\hat{N}_K$ the vector of $I\!R^{NA_i}$ defined by

$$\text{for } a, a' \in A_i, \quad \hat{L}_{K,aa'} = L_{K,ee'} \text{ if } a = e, \quad a' = e' \text{ in } K$$

$$\text{for } a \in A_i, \ \hat{N}_{K,a} = N_{K,e} \text{ if } a = e \text{ in } K$$

We define also $\hat{L}_{K,a}$ the $NA_i \times NA_i$ matrix whose non-zero coefficients are on the line number a in the matrix $\hat{L}_K$. The relation (25_1) is equivalent to

$$\hat{L}_{K_1,a}.U_i + \hat{L}_{K_2,a}.U_i = \hat{N}_{K_1,a} + \hat{N}_{K_2,a} \text{ for } a \in A_i$$

hence

$$\mathcal{A}_i = \sum_{a \in A_i} \hat{L}_{K_1,a} + \hat{L}_{K_2,a} = \sum_{K \in \mathcal{T}_h} \hat{L}_K$$

Since L_K is symmetric, so is $\hat{L}_K$, hence $\mathcal{A}_i$ is also symmetric. Moreover the following relation holds for each $V \in I\!R^{NA_i}$

$$V^T \mathcal{A}_i V = \sum_{K \in \mathcal{T}_h} V^T \hat{L}_K V = \sum_{K \in \mathcal{T}_h} V_K^T L_K V_K$$

Because of the positiveness of L_K, we have $V^T \mathcal{A}_i V \geq 0$. The definiteness of $\mathcal{A}_i$ results of the uniqueness result of the theorem 1. ■

4.2. Effective order of the scheme

In order to check the second order accuracy of the scheme, we have performed simple tests on the Poisson problem on the square $\Omega = [0,1]^2$. We solve a problem

$$\begin{aligned} -\Delta u &= f_k \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

where $f_k(x,y) = ((2\pi k_1)^2 + (2\pi k_2)^2)\sin\, 2\pi k_1 x\, \sin\, 2\pi k_2 y$. For different values of $k = (k_1, k_2)$. The exact solution is $u_k(x,y) = \sin(2\pi k_1 x)\sin(2\pi k_2 y)$. We use four meshes with respectively 100, 400, 1600, 3600 triangles. The mesh $\mathcal{T}_h$ is a regular triangulation consisting on squares divided in 4 triangles. The parameter h is the length of the edge of the square. The Table 1 reports the values of $|u - u_h|_{0,\Omega}$ for (k_1, k_2) =(1,1), (3,3), (15,15), (30,30). In this latest case, the finest mesh (3600 triangles) should have the limit resolution (one period for h). On Figure 4, we have plotted in Log-Log scale the points of the Table 1.

	$h = 0.2$	$h = 0.1$	$h = 0.05$	$h = 0.0333$
$(k_1, k_2) = (1,1)$	$2.63\ 10^{-2}$	$6.57\ 10^{-3}$	$1.64\ 10^{-3}$	$7.30\ 10^{-4}$
$(k_1, k_2) = (3,3)$	0.237	$5.92\ 10^{-2}$	$1.48\ 10^{-2}$	$6.57\ 10^{-3}$
$(k_1, k_2) = (15,15)$	2.271	4.590	0.3737	0.165
$(k_1, k_2) = (30,30)$	1.633	2.271	4.590	0.261

Table 1 : *Value of the error* $|u - u_h|_{0,\Omega}$ *for different meshes and different solutions*

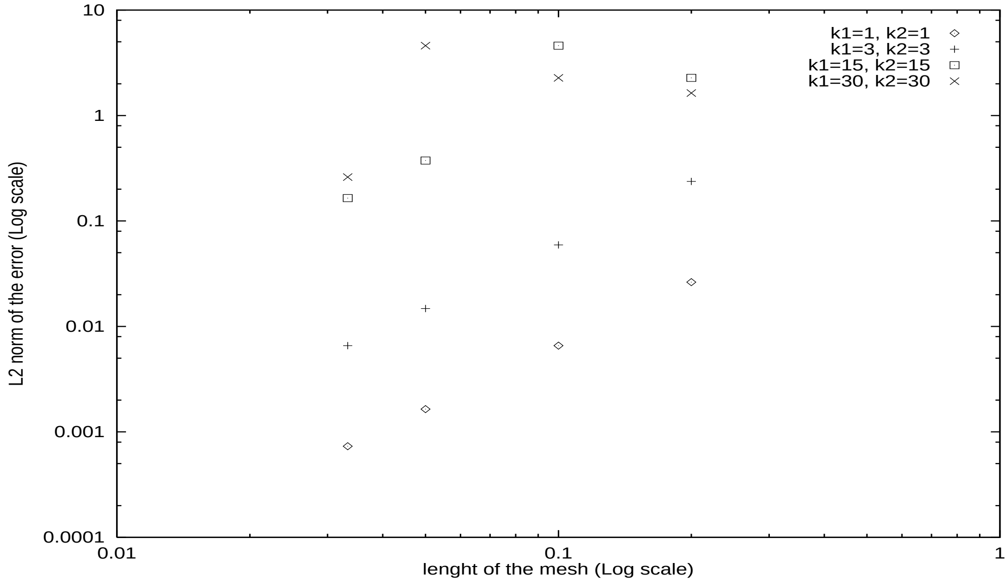

Fig. 4. $|u - u_h|_{0,\Omega}$ *versus* h *in Log scale*

As expected, the slope of the line are 2 for the “low frequence” solutions $(k_1, k_2) =$ (1,1) or (3,3). For (k_1, k_2) =(15,15) , the convergence begins only with the two finest meshes, whereas it it not really reached for $(k_1, k_2) = (30, 30)$.

4.3. A singular test case

This test-case, proposed by Johnson in [23], is to find the solution of

$$\begin{cases} -\Delta u = 0 & \text{on } \Omega = [-1, 1]^2 \\ u = g & \text{on } \partial\Omega \end{cases} \tag{26}$$

which exact solution is $u(x, y) = \text{arc tan}\left(\frac{y}{x+1}\right)$. The boundary condition is $g(x, y) = u(x, y)/\partial\Omega$. The solution has a singularity at $(-1, 0)$. On Figure 5 are displayed the exact solution, the computed solution and the L^∞ error on a mesh of 400 triangles. This test is interesting because $u \notin H^1$. As expected, the error is $O(1)$ at the singularity. Note the continuity of u_h at the mid-edge points.

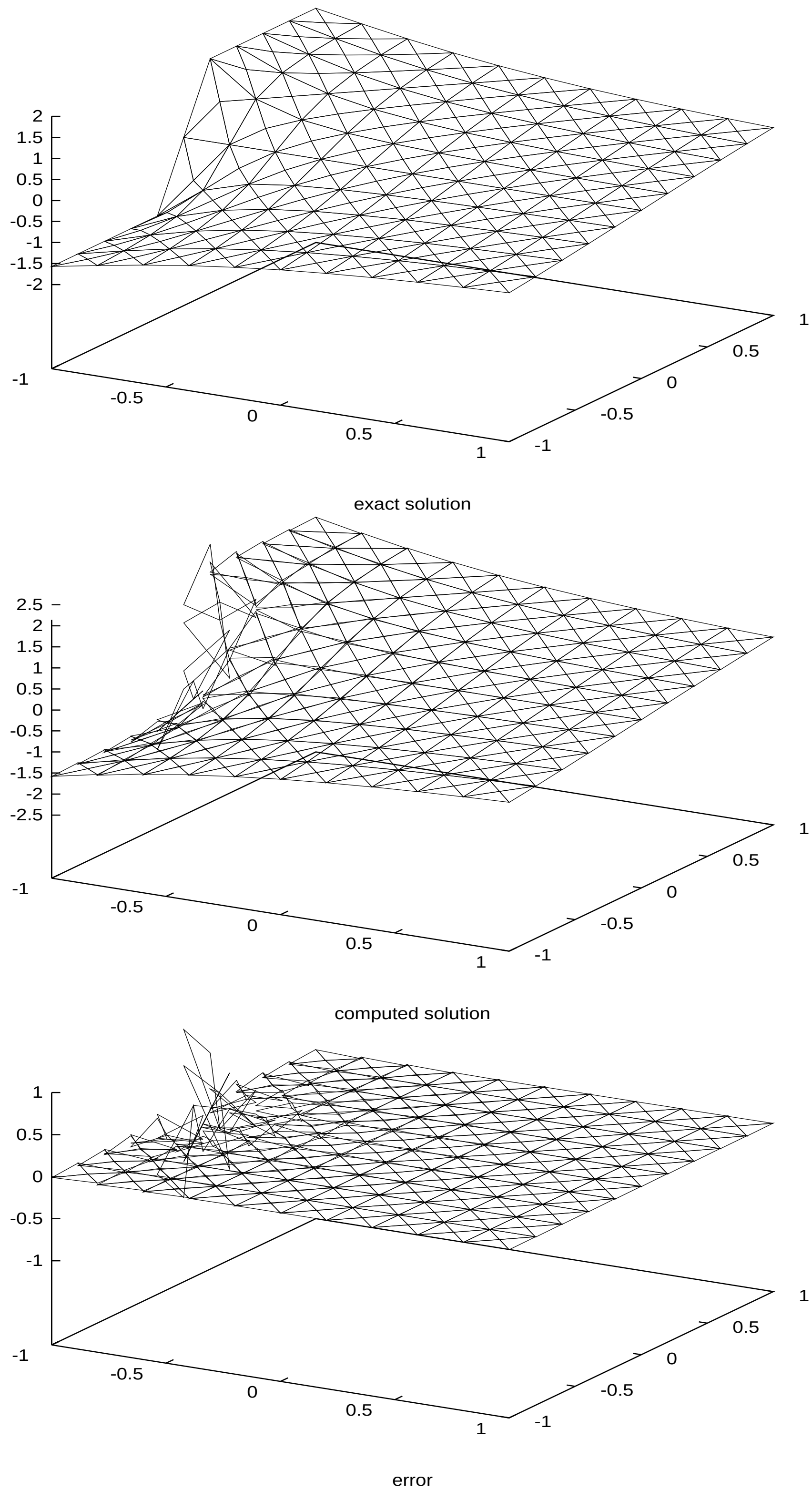

Fig. 5. *Exact, computed solution and L^∞ error on the test case of Johnson. (400 triangles).*

5. Conclusion

We present in this paper a finite volume scheme apparently new, which is a generalization to triangular meshes of the Keller's box scheme. The framework of the finite element spaces is used systematically and allows to prove an estimation error in the discrete energy norm for the Poisson problem.

The main feature of this scheme is that, as in the original box-scheme of Keller [17], piecewise linear spaces are used both for the solution and the fluxes (the gradient). This aspect seems particularly suited for complex elliptic problems. Moreover, the extension of this scheme to 3-dimensional computations on tetrahedral meshes is straightforward. Note finally that the evolutive version of the scheme is implicit. This appears to be particularly interesting for complex parabolic problems where large time steps can be used.

Objective explored in a near future are :

1. A careful comparison with the standard mixed finite element method has to be carried out, especially for problems with large variation of the diffusion coefficients within a cell. Typical examples are boundary layers computations. The Stokes problem can also be an interesting test comparison.
2. Parabolic problems involving complex fluxes.
3. The compressible Navier-Stokes equations. The introduction of upwinding in box schemes for compressible flows have already been explored in [7,8,9,24,25] and requires further developments.

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