Some nonconforming mixed box schemes for elliptic problems

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In this paper, we introduce three schemes for the Poisson problem in 2D on triangular meshes, generalizing the FVbox scheme introduced in [10]. In this kind of scheme, the approximation is performed on the mixed form of the problem, but contrary to the standard mixed method, with a pair of trial spaces different from the pair of test spaces. The latter is made of Galerkin-discontinuous spaces on an unique mesh. The first scheme uses as trial spaces the P^1 nonconforming space of Crouzeix-Raviart both for u and for the flux $p = \nabla u$. In the two others the quadratic nonconforming space of Fortin and Soulie is used. An important feature of all these schemes is that they are equivalent to a first scheme in u only and an explicit representation formula for the flux $p = \nabla u$. The numerical analysis of the schemes is performed using this property.

I. INTRODUCTION

The aim of this paper is to introduce three box schemes for elliptic problems, based on the model of [10, 11]. This class of schemes belongs to the category of so called *mixed Petrov-Galerkin methods*. This means that a mixed form of the problem is discretized with trial functions different from the test functions. The interest of these schemes is that they combine the advantages of the finite volumes and of the finite element mixed methods within a simple framework. Their main features are:
- the two equations of the mixed form are discretised at the level of the same mesh, by simply averaging (Finite Volume Methods). One speaks of "compact schemes".
- standard mixed finite elements for trial spaces are used. The test functions are of Galerkin-discontinuous type.
- these schemes can be seen as a natural generalisation of Keller's box scheme [19] (interface degrees of freedom for u and ∇u).

In the sequel, we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with regular or convex boundary in order for the homogeneous Poisson problem to be well-posed in $H_0^1(\Omega) \cap H^2(\Omega)$ with data in $L^2(\Omega)$. The mixed form we consider is: find $(u, p) \in M_1 \times X_1$ such that

$$\begin{cases} (\operatorname{div} p + f, v)_{0,\Omega} = 0 & \forall v \in M_2 \\ (p - \nabla u, q)_{0,\Omega} = 0 & \forall q \in X_2 \end{cases} \tag{1.1}$$

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The standard setting holds with $M_1 \times X_1 = H_0^1 \times H_{\rm div}$ (or $H_0^1 \times (H^1)^2$ in Sect. 2) and $M_2 \times X_2 = L^2 \times (L^2)^2$. Recall that the divergence conforming space is $H_{\rm div}(\Omega) = \{p \in (L^2(\Omega))^2 / \operatorname{div} p \in L^2(\Omega)\}$. We denote $|\cdot|_{m,\Omega}$, $\|\ \ \|_{m,\Omega}$ the m-semi norm and the m norm on $H^m(\Omega)$. In addition, we note $\|p\|_{\rm div,\Omega} = (|p|^2_{0,\Omega} + |\operatorname{div} p|^2_{0,\Omega})^{1/2}$ and $|p|_{\rm div,\Omega} = |\operatorname{div} p|_{0,\Omega}$. The approximation of (1.1) on a regular mesh $\mathcal{T}_h$ made of triangles K is performed by the scheme

$$\begin{cases} \displaystyle\sum_{K\in\mathcal{T}_h} (\operatorname{div} p_h + f, v_h)_{0,K} = 0 \quad \forall v_h \in M_{2,h} \\ \displaystyle\sum_{K\in\mathcal{T}_h} (p_h - \nabla u_h, q_h)_{0,K} = 0 \quad \forall q_h \in X_{2,h} \end{cases} \tag{1.2}$$

In [10, 11], the scheme (1.2) is addressed with the discrete spaces $(M_{1,h}, X_{1,h}) = (P^1_{nc,0}, RT^0)$ (P^1 nonconforming space of Crouzeix-Raviart and divergence conforming space of Raviart-Thomas of lowest order). Note that this scheme has also been addressed in [21] , and that it coincides with an hybrid form of the standard mixed method of Raviart-Thomas ([2, 20]).

In this paper, we are going to describe schemes of type (1.2) with discrete spaces $M_{1,h}$, $X_{1,h}$ each of one being non-conforming in H_0^1 , $H_{\rm div}$. The interest of this class of schemes is that they use trial spaces of higher order than the standard Galerkin mixed methods, due to the fact the the Babuška-Brezzi condition takes place between $M_{1,h} \times X_{1,h}$ and $M_{2,h} \times X_{2,h}$ and not between $M_{1,h}$ and $X_{1,h}$. In addition, contrary to many other schemes combining the mixed finite element and the finite volume formalism [4, 17, 22, 25], only one mesh is used. The price to pay for that advantage is the difficulty to find spaces $M_{1,h} \times X_{1,h}$ and $M_{2,h} \times X_{2,h}$ coupled by a stability condition. A minimal request with this respect is of course that $\dim M_{1,h} + \dim X_{1,h} = \dim M_{2,h} + \dim X_{2,h}$.

The numerical analysis of (1.2) can be performed in two ways. The first one is proving the Babuška-Brezzi stability conditions [3, 7, 22, 5] for the underlying bilinear form B_h on $(M_{1,h} \times X_{1,h}) \times (M_{2,h} \times X_{2,h})$ defined by

$$B_h\left[(u_h\,,\,p_h)\;;\;(v_h\,,\,q_h)\right] = (p_h\,,\,q_h)_{0,\Omega} - \sum_K (\nabla\, u_h\,,\,q_h)_{0,K} + \sum_K (\operatorname{div}\,p_h\,,\,v_h)_{0,K} \tag{1.3}$$

The other possibility we follow throughout this paper, is to express the flux p_h in function of u_h and f by a purely local formula. Therefore, our schemes are equivalent to a non-conforming scheme in u_h and a local reconstruction formula of p_h in function of u_h. Note that the non-conforming methods have been introduced in works such as [12], [24]. They have recently gained a renewal of interest ([18], [13]) especially as a posteriori interpretations of mixed hybridized methods. That topics is systematically explored in [1], where a broad number of standard mixed methods is proved to be equivalent to a non-conforming method in the principal unknow and a local reconstruction of the flux. However, the situation is much more simpler in our schemes since no hybridization is required to get this equivalence.

Let us give now some standard notations. We introduce the mesh dependent spaces $(H_0^1 + M_{1,h}), (H_{\rm div} + X_{1,h})$, equipped with the mesh dependent norms

$$|u|_{1,h} = \Big(\sum_K |\nabla\, u|^2_{0,K}\Big)^{1/2} \;\;;\;\; \|u\|_{1,h} = (|u|^2_{0,\Omega} + |u|^2_{1,h})^{1/2} \quad , \quad u \in H_0^1 + M_{1,h} \tag{1.4}$$

$$|p|_{\mathrm{div},h} = \Big(\sum_K |\operatorname{div} p|^2_{0,K}\Big)^{1/2} \;\; ; \;\; \|p\|_{\mathrm{div},h} = (|p|^2_{0,\Omega} + |p|^2_{\mathrm{div},h})^{1/2},\, p \in H_{\mathrm{div}} + X_{1,h} \quad (1.5)$$

We denote also $H_{1,h} = (H^1_0 + M_{1,h}) \times (H_{\mathrm{div}} + X_{1,h})$, $H_{2,h} = L^2 \times (L^2)^2$ and by $\| \;\; \|_{H_{1,h}}$, $\| \;\; \|_{H_{2,h}}$ the norms

$$\|(u,p)\|_{H_{1,h}} = (\|u\|^2_{1,h} + \|p\|^2_{\mathrm{div},h})^{1/2} \;\; ; \;\; \|(v,q)\|_{H_{2,h}} = (|v|^2_{0,\Omega} + |q|^2_{0,\Omega})^{1/2} \quad (1.6)$$

B_h is the bilinear continuous form onto $H_{1,h} \times H_{2,h}$

$$B_h\,[(u,p)\;;\;(v,q)] = (p,q)_{0,\Omega} - \sum_K (\nabla u\,,\,q)_{0,K} + \sum_K (\operatorname{div} p\,,\,v)_{0,K} \quad (1.7)$$

We consider a regular finite element triangular mesh $\mathcal{T}_h$ in the usual sense. The geometrical notation is as follows. The triangles are denoted by K. We denote by $\partial K = \{e, e', e''\}$ the edges of K . The edges e, e', e'' are opposite to the vertices S, S', S'' and to the angles $\theta, \theta', \theta''$. The barycentric coordinates corresponding to S, S', S'' are $\lambda_S, \lambda_{S'}, \lambda_{S''}$. The outgoing unitary normal vector to e of each tringle K is $\nu_{K,e}$, the corresponding tangential vector is $\tau_{K,e} = R_{+\frac{\pi}{2}}(\nu_{K,e})$, where $R_{+\frac{\pi}{2}}$is the positive rotation of angle $\frac{\pi}{2}$. The barycenter of K is x_K and the midedge of e is x_e. We note $a \in \mathcal{A} = \mathcal{A}_i \cup \mathcal{A}_b$ the edges with global numbering. The sets $\mathcal{A}_i$ (resp. $\mathcal{A}_b$) denote the internal (resp. boundary) edges. The number of triangles is NE. The number of edges (resp. internal, boundary) is NA (resp. NA_i, NA_b). The number of vertices (resp. internal, boundary) is NV (resp. NV_i, NV_b). The Euler relations write

$$3NE + NA_b = 2NA \quad (1.8)$$

$$NE - NA + NV = 1 \quad (1.9)$$

The gradient of f is $\nabla f = [\partial_x f\,,\,\partial_y f]$ and the 2D rotational is $\nabla^{\perp} f = [\partial_y f\,,\,-\partial_x f] = R_{-\frac{\pi}{2}}(\nabla f)$.

The outline of the paper is as follows. In Sect.2, we perform the numerical analysis of scheme (1.2) where $M_{1,h} = P^1_{nc,0}$ and $X_{1,h}$ is the space of P^1 vector fields in each triangle with continuity of the normal component at the middle of each edge. Although it is nonconforming in H_{div}, this space is very close of the classical Raviart-Thomas space RT^0. After elimination of p_h, we obtain a scheme in u_h only, very close from the one obtained in [11]. More precisely, the source term is of the form $(f, \tilde{v}_h)$, where $\tilde{v}_h$ is a non standard modification of the test function v_h. This scheme has been introduced by B. Courbet in [9]. The second scheme is based onto the non-conforming piecewise quadratic $P^2_{nc,0}$ space of Fortin and Soulie [15] for u_h. Our scheme allows to give a natural interpretation both of the method in [15] and of its mixed interpretation given in [14]. The third scheme is the natural extension to the quadratic case of the affine version of the FVbox scheme [10, 11]. Again it is based onto the $P^2_{nc,0}$ space and onto the Raviart-Thomas space RT^1 for the flux [23]. Optimal order estimates are derived in each case. To each of these three schemes, corresponds a dual scheme obtained by inverting trial and test functions. These dual schemes are non standard and are not further studied in this paper. However, they have their own interest.

Although we limit ourselves to the academic Poisson problem in order to preserve clarity, these schemes can handle clearly more complex closure laws for the flux p or Neumann boundary conditions.

II. THE CASE $\boldsymbol{P^1_{NC,0}} \times [(\boldsymbol{P^1_{NC}}) \times (\boldsymbol{P^1_{TD}})]$

A. Discrete equations

The first scheme we study is based on [9], where the degrees of freedom of u_h and p_h and the discrete equations are introduced. Here, we perform the numerical analysis of the scheme in the framework of a mixed Petrov-Galerkin method. The trial space for u_h is $P^1_{nc,0} = M_{1,h}$, (non-conforming homogeneous space of Crouzeix-Raviart [12]) defined by

$$\begin{aligned} P^1_{nc,0} \;=\; & \{v_h \,/\, \forall\, K \in \mathcal{T}_h,\;\; v_{h|K} \in P^1(K), v_h \text{ is continuous at the middle} \\ & \text{of each edge},\; v_h = 0 \text{ at the middle of each edge on } \partial\Omega\} \end{aligned}$$

The local degrees of freedom of $P^1_{nc,0}$ are the linear forms $< l_{K,e}\ ;\ u_{h|K} >= u_h\,(x_e)$ and the local basis is

$$p_{K,e}(x) = 1 - 2\lambda_S\,(x) \tag{2.1}$$

The decomposition of $u_{h|K}$ in this basis is

$$u_{h|K}(x) = \sum_{e\in\partial K} u_{K,e}\, p_{K,e}(x) \tag{2.2}$$

The global degrees of freedom are the forms $< l_a\ ;\ u_h >= u_h\,(x_a)\;\; a \in \mathcal{A}_i$. The global basis of the Crouzeix-Raviart space $P^1_{nc,0}$ is

$$p_a(x) = p_{K_1,e_1}\,(x)\; 1\!\!1_{\bar{K}_1}\,(x) + p_{K_2,e_2}\; 1\!\!1_{\overset{\circ}{K}_2}\,(x) \tag{2.3}$$

We take $\bar{K}_1$ and $\overset{\circ}{K}_2$ in order to have $p_{a|a} = 1$. The edge a is oriented from K_1 toward K_2, with $a = e_1$ in K_1 and $a = e_2$ in K_2. The global decomposition of u_h is

$$u_h(x) = \sum_{a\in\mathcal{A}_i} u_a\, p_a(x) \tag{2.4}$$

with $u_a = u_{K_1,e_1} = u_{K_2,e_2}$. The space $X_{1,h}$ is

$$X_{1,h} \;=\; \Big\{p_h\ :\ \Omega \to \mathbb{R}^2 \,/\, p_{h|K} \in P^1(K)^2 \quad \forall\, K\,,\, p_h\cdot\nu_a \tag{2.5}$$

$$\text{continuous at the middle of each edge } a \in A_i\Big\} \tag{2.6}$$

The local degrees of freedom of $X_{1,h}$ are

$$< L_{K,e}\ ;\ p_{h|K} >= \int_e p_h \cdot \nu_{K,e} \quad ; \quad < M_{K,e}\ ;\ p_{h|K} >= \int_e p_h \cdot \tau_{K,e} \tag{2.7}$$

The corresponding local basis is $\mathcal{B}_K = \Big\{P_{K,e}\,(x)\,,\, Q_{K,e}\,(x)\,,\, e \in \partial\, K\Big\}$ where the affine vectorial functions $P_{K,e}$, $Q_{K,e}$ are

$$P_{K,e}\,(x) = \frac{p_{K,e}\,(x)}{|e|}\,\nu_{K,e} \qquad ; \qquad Q_{K,e}\,(x) = \frac{p_{K,e}\,(x)}{|e|}\,\tau_{K,e} \tag{2.8}$$

$p_{h|K}$ may be written in the form

$$p_{h|K}\,(x) = \sum_{e\in\partial K} [v_{K,e}\, P_{K,e}\,(x) + w_{K,e}\, Q_{K,e}\,(x)] \tag{2.9}$$

with $v_{K,e} =< L_{K,e}\ ;\ p_{h|K} >$, $w_{K,e} =< M_{K,e}\ ;\ p_{h|K} >$. The global degrees of freedom of $X_{1,h}$ are the linear forms defined for $a \in \mathcal{A}$, $K \in \mathcal{T}_h, e \in \partial K$

$$< L_a\,;p_h >= \int_a p_h \cdot \nu_a \qquad ; \qquad < M_{K,e}\ ;\ p_h >=< M_{K,e}\ ;\ p_{h|K} > \tag{2.10}$$

The corresponding global basis is $\mathcal{B} = \{P_a(x)\,,\, Q_{K,e}(x)\,,\, a\ \in \mathcal{A}\,,\, K \in \mathcal{T}_h\,,\, e \in \partial K\}$ with $P_a\,(x) = \frac{1}{|a|}\,p_a\,(x)\,\nu_a$. The global decomposition of $p_h \in X_{1,h}$ is

$$p_h(x) = \sum_{a\in A} v_a\,P_a\,(x) + \sum_K \sum_{e\in\partial K} w_{K,e}\,Q_{K,e}(x) \tag{2.11}$$

where $v_a =< L_a\,;p_h >$. In addition $v_a = v_{K_1,e_1} = -v_{K_2,e_2}$. Let us describe now the discrete system (1.2). Equation $(1.2)_a$ is equivalent to the NE equations

$$\sum_{e\in\partial K} v_{K,e} = -|K|\,(\Pi^0\,f)_K \tag{2.12}$$

where Π^0 is the orthogonal projector onto the space P^0 of the constant functions in each triangle. Equation $(1.2)_b$ yields the $2NE$ equations

$$0 = \int_K (p_h - \nabla\,u_h)\ \ ,\ \ \forall K \in \mathcal{T}_h \tag{2.13}$$

or equivalently

$$\frac{|K|}{3} \sum_{e\in\partial K} \Big[v_{K,e}\,\frac{1}{|e|}\,\nu_{K,e} + w_{K,e}\,\frac{1}{|e|}\,\tau_{K,e}\Big] = \sum_{e\in\partial K} u_{K,e}\,\frac{|e|}{|K|}\,\nu_{K,e} \tag{2.14}$$

If we impose the homogeneous Dirichlet boundary conditions by

$$u_a = 0 \qquad \forall\, a \in \mathcal{A}_b \tag{2.15}$$

then we have $3NE + NA_b$ equations. On the other side the total number of unknows is $2NA + 3NE$. Since $3NE + NA_b = 2NA$, we have to find $3NE$ additional equations. The choice suggested in [9] is to enforce locally the equations $p_h - \nabla\,u_h = 0$ along each edge e of each triangle K by :

$$\int_e (p_h - \nabla\,u_h)\cdot\tau_{K,e} = 0 \quad \forall K \in \mathcal{T}_h\,,\, \forall e \in \partial K \tag{2.16}$$

or equivalently

$$w_{K,e} = 2\,(u_{K,e'} - u_{K,e''}) \tag{2.17}$$

This is equivalent to write the $3NE$ orthogonality relations

$$\int_K (p_h - \nabla\,u_h)\cdot Q_{K,e} = 0 \qquad \forall K \in \mathcal{T}_h\,,\, e \in \partial K \tag{2.18}$$

This suggests to select the following test spaces $X_{2,h}$ and $M_{2,h}$ in (1.2)

- $X_{2,h} = (P^0)^2 + Vect\,\{Q_{K,e}\,,\, K \in \mathcal{T}_h\,,\, e \in \partial\,K\}$
- $M_{2,h} = P^0$

The dimensions of $X_{2,h}$ and $M_{2,h}$ are

$$\dim\,X_{2,h} = 5NE \quad \dim\,M_{2,h} = NE \tag{2.19}$$

Therefore, due to (1.8), the following necessary relation holds

$$\dim X_{1,h} + \dim M_{1,h} = \dim X_{2,h} + \dim M_{2,h} \tag{2.20}$$

Note that $X_{2,h}$ is nothing but the space

$$X_{2,h} = \left\{ q_h \,:\, \Omega \to \mathbb{R}^2 \,/\, q_{h|K} \in P^1\,(K)^2 \,,\ \mathrm{div}\ q_{h|K} = 0 \quad \forall\, K \in \mathcal{T}_h \right\} \tag{2.21}$$

This results from the fact that for any $K \in \mathcal{T}_h$, $e \in \partial K$, $|K|\,\mathrm{div}\ Q_{K,e} = \nu_{K,e} \cdot \tau_{K,e} = 0$ and from a dimensional comparison. Finally, the discrete system can be settled as: find $(u_h\,,\,p_h) \in M_{1,h} \times X_{1,h}$ such that

$$\left\{ \begin{array}{l} \displaystyle\sum_K (\mathrm{div}\, p_h + f, v_h)_{0,K} = 0 \quad \forall v_h \in M_{2,h} \\ \displaystyle\sum_K (p_h - \nabla u_h, q_h)_{0,K} = 0 \quad \forall q_h \in X_{2,h} \end{array} \right. \tag{2.22}$$

or equivalently, for each $K \in \mathcal{T}_h$

$$\left\{ \begin{array}{ll} \displaystyle\sum_{e \in \partial K} v_{K,e} = -|K|\ (\Pi^0 f)_K & NE \text{ equations} \\ \displaystyle\frac{1}{3} \sum_{e \in \partial K} \left[\frac{v_{K,e}}{|e|}\, \nu_{K,e} + \frac{w_{K,e}}{|e|}\, \tau_{K,e} \right] = \sum_{e \in \partial K} u_{K,e}\, \frac{|e|}{|K|}\, \nu_{K,e} & 2\ NE \text{ equations} \\ w_{K,e} = 2\,(u_{K,e'} - u_{K,e''}) & 3\ NE \text{ equations} \end{array} \right. \tag{2.23}$$

with the boundary conditions $u_a = 0$ for $a \in A_b$.

Let us briefly describe now the matrix form of (2.23), (cf also [10]). Introducing the notation

$$U_K = \begin{bmatrix} u_{K,e_1} \\ u_{K,e_2} \\ u_{K,e_3} \end{bmatrix} \quad ; \quad V_K = \begin{bmatrix} v_{K,e_1} \\ v_{K,e_2} \\ v_{K,e_3} \end{bmatrix} \quad ; \quad W_K = \begin{bmatrix} w_{K,e_1} \\ w_{K,e_2} \\ w_{K,e_3} \end{bmatrix} \tag{2.24}$$

(2.23) may be written as

$$\left\{ \begin{array}{c} -\tilde{L}_{K,1}\ U_K + \tilde{M}_{K,1}\, V_K + \tilde{M}_{K,2}\, W_K = -\tilde{N}_K \\ W_K = \tilde{L}_{K,2}\, U_K \end{array} \right. \tag{2.25}$$

with $\nu_i = \nu_{K,e_i}$, $\tau_i = \tau_{K,e_i}$, $i = 1, 2, 3$,

$$\tilde{L}_{K,1} = \frac{1}{|K|} \begin{bmatrix} 0 & 0 & 0 \\ |e_1|\,\nu_1^x & |e_2|\,\nu_2^x & |e_3|\,\nu_3^x \\ |e_1|\,\nu_1^y & |e_2|\,\nu_2^y & |e_3|\,\nu_3^y \end{bmatrix} \quad ; \quad \tilde{M}_{K,1} = \frac{1}{3|K|} \begin{bmatrix} 3 & 3 & 3 \\ \frac{|K|}{|e_1|}\,\nu_1^x & \frac{|K|}{|e_2|}\,\nu_2^x & \frac{|K|}{|e_3|}\,\nu_3^x \\ \frac{|K|}{|e_1|}\,\nu_1^y & \frac{|K|}{|e_2|}\,\nu_2^y & \frac{|K|}{|e_3|}\,\nu_3^y \end{bmatrix}$$

$$\tilde{M}_{K,2} = \frac{1}{3|K|} \begin{bmatrix} 0 & 0 & 0 \\ \frac{|K|}{|e_1|}\,\tau_1^x & \frac{|K|}{|e_2|}\,\tau_2^x & \frac{|K|}{|e_3|}\,\tau_3^x \\ \frac{|K|}{|e_1|}\,\tau_1^y & \frac{|K|}{|e_2|}\,\tau_2^y & \frac{|K|}{|e_3|}\,\tau_3^y \end{bmatrix}$$

$$\tilde{N}_k = \begin{bmatrix} (\Pi^0 f)_K \\ 0 \\ 0 \end{bmatrix} \quad ; \quad \tilde{L}_{K,2} = 2 \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Equation $(2.25)_a$ gives V_K in function of U_K by

$$V_K = -N_K + L_{K,1}\, U_K \tag{2.26}$$

with

$$N_K = \tilde{M}_{K,1}^{-1}\, \tilde{N}_k \quad ; \quad L_{K,1} = \tilde{M}_{K,1}^{-1}\left[\tilde{L}_{K,1} - \tilde{M}_{K,2}\, \tilde{L}_{K,2}\right] U_K \tag{2.27}$$

System (2.23) results now in a system in the $(u_a)_{a\in A_i}$ unknowns by eliminating v_a for any internal edge $a \in \mathcal{A}_i$, with orientation from K_1 towards K_2 , $a = e_1$ in K_1 , $a = e_2$ in K_2 by $v_{K_1,e_1} + v_{K_2,e_2} = 0$. This yields a linear system $AU = b$, $U = (u_a)_{a\in\mathcal{A}_i}$, which is

$$[L_{K,1}\; U_{K_1}]_{e_1} + [L_{K,2}\; U_{K_2}]_{e_2} = [N_{K_1}]_{e_1} + [N_{K_2}]_{e_2} \tag{2.28}$$

Once (2.28) is solved, $p_{h|K}$ is evaluated by (2.9) where V_K and W_K are given by (2.26), (2.25). Note finally that (2.9) is nothing but the local following Helmoltz decomposition

$$p_{h|K}(x) = \nabla\, \varphi_h^1 + \nabla^\perp\, \varphi_h^2 \tag{2.29}$$

where

$$\varphi_h^1\,(x) = \frac{|K|}{2} \sum_{e\in\partial K} \frac{v_{K,e}}{|e|^2}\, p_{K,e}^2\,(x) \;\; ; \;\; \varphi_h^2\,(x) = -\frac{|K|}{2} \sum_{e\in\partial K} \frac{w_{K,e}}{|e|^2}\, p_{K,e}^2\,(x) \tag{2.30}$$

B. Numerical analysis

Lemma 2.1. *Discrete system* (2.23) *has an unique solution.*

Proof. Since dim $X_{1,h}$ + dim $M_{1,h}$ = dim $X_{2,h}$ + dim $M_{2,h}$, it is sufficient to prove that $(\Pi^0 f)_K = 0$ for any K ensures $u_h = 0$, $p_h = 0$. From $(\text{div}\; p_h\,,\, v_h)_{0,K} = 0$ for any $v_h \in P^0$, we infer by (2.21) that $p_h \in X_{2,h}$. Taking $q_h = p_h - \nabla u_h$ in (2.22), we deduce that $p_h = \nabla u_h$. In particular p_h is constant in each triangle K. We have now

$$\begin{aligned}\sum_K (p_h\,,\, \nabla u_h)_{0,K} &= -\sum_K (\text{div}\; p_h\,,\, u_h)_{0,K} + \sum_K \sum_{e\in\partial K} \int_e (p_h \cdot \nu_{K,e})\, u_h \\ &= \sum_K \Big[\sum_{e\in\partial K} (p_h \cdot \nu_{K,e}) \int_e (u_h - u_{K,e}) + \sum_{e\in\partial K} u_{K,e} \int_e p_h \cdot \nu_{K,e} \Big]\end{aligned}$$

Since $u_{K,e} = \dfrac{1}{|e|} \displaystyle\int_e u_h$ and

$$\sum_K \sum_{e\in\partial K} u_{K,e} \int_e p_h \cdot \nu_{K,e} = \sum_{a\in A_i} u_a \Big[\int_a (p_{h|K_1} \cdot \nu_a - p_{h|K_2} \cdot \nu_a) \Big]$$

we deduce by midpoint continuity of $p_h \cdot \nu_a$ on each interior edge a,

$$0 = \sum_K (p_h, \nabla u_h)_{0,K} = \sum_K |p_h|_{0,K}^2 = \sum_K |\nabla u_h|_{0,K}^2$$

Therefore $p_h = \nabla u_h = 0$ and then $u_h = 0$.

∎

In order to obtain well-posedness of discrete problem (2.22), let us explicitely perform the elimination of p_h in order to write the scheme in the following form

- a discrete system in u_h only.
- a local reconstruction formula of p_h in function of u_h.

Proposition 2.2. *The FVbox scheme* (2.22) *is equivalent to the following scheme :*
(i) $u_h \in P^1_{nc,0}$ *is the solution of the modified non-conforming method*

$$\sum_K (\nabla u_h \,,\, \nabla v_h)_{0,K} = \sum_K (f,\, \Pi^0 v_h + \nabla v_h \cdot A_K)_{0,K} \qquad \forall\, v_h \in P^1_{nc,0} \tag{2.31}$$

where A_K *is the vector depending only of the geometry of* K*, given by*

$$A_K = \frac{1}{6\sum_{\bar e} |\bar e|^2} \sum_{e \in \partial K} (|e'|^2 - |e''|^2)\, |e|\, \tau_{K,e} \tag{2.32}$$

(ii) The piecewise affine function p_h *is given in function of* ∇u_h *and* $\Pi^0 f$ *by the local representation formula of* p_h *in function of* ∇u_h *and* $\Pi^0 f$ *is*

$$p_{h|K}\,(x) = \nabla\, u_{h|K} + \nabla\, p_{h|K} \cdot \overrightarrow{x_K\, x} \tag{2.33}$$

where

$$\nabla\, p_{h|K} = -\frac{(\Pi^0 f)_K}{\sum\limits_{\bar e \in \partial K} |\bar e|^2} \sum_{e \in \partial K} |e|^2\, \nu_e \otimes \nu_e \tag{2.34}$$

(iii) The corresponding local decomposition (2.9) *of* $p_{h|K}$ *is*

$$p_{h|K}\,(x) = \sum_{e \in \partial K} [v_{K,e}\, P_{K,e}\,(x) + w_{K,e}\, Q_{K,e}\,(x)] \tag{2.35}$$

with

$$v_{K,e} = |e|\, \frac{\partial u_h}{\partial \nu_{K,e}} - \frac{|e|^2 |K|\, (\Pi^0 f)_K}{\sum_{\bar e \in \partial K} |\bar e|^2} \quad , \quad w_{K,e} = |e|\, \frac{\partial u_h}{\partial \tau_{K,e}} \tag{2.36}$$

Proof. Let us first prove *(iii)* . Taking the average of $p_{K,e}\,(x)$ over K, we get since, $\int_e p_{K,e}\,(x) = \frac{|K|}{3}$

$$\frac{1}{|K|} \int_K p_h = \frac{1}{3} \sum_{e \in \partial K} \frac{1}{|e|}\, [v_{K,e}\, \nu_{K,e} + w_{K,e}\, \tau_{K,e}] = \nabla\, u_{h|K} \tag{2.37}$$

For each $e \in \partial K$, the decomposition of $\nabla\, u_{h|K}$ in $(\nu_{K,e}\,,\, \tau_{K,e})$ is

$$\nabla\, u_{h|K} = \frac{\partial u_h}{\partial \nu_{K,e}}\, \nu_{K,e} + \frac{\partial u_h}{\partial \tau_{K,e}}\, \tau_{K,e} \tag{2.38}$$

Therefore we have still

$$\nabla\, u_{h|K} = \frac{1}{3} \sum_{e \in \partial K} [\frac{\partial u_h}{\partial \nu_{K,e}}\, \nu_{K,e} + \frac{\partial u_h}{\partial \tau_{K,e}}\, \tau_{K,e}] \tag{2.39}$$

Identifying (2.37) and (2.39) and taking in account $w_{K,e} = |e|\, \frac{\partial u_h}{\partial \tau_{K,e}}$ we obtain

$$\sum_{e \in \partial K} \frac{1}{|e|}\, v_{K,e}\, \nu_{K,e} = \sum_{e \in \partial K} \frac{\partial u_h}{\partial \nu_{K,e}}\, \nu_{K,e} \tag{2.40}$$

From the identity $\sum_{e\in\partial K} |e|\,\nu_{K,e} = 0$, we deduce that there exists a constant C_K such that

$$\frac{v_{K,e}}{|e|^2} = \frac{1}{|e|}\,\frac{\partial u_h}{\partial \nu_{K,e}} + C_K \tag{2.41}$$

Finally, since $\sum_{e\in\partial K} v_{K,e} = -|K|\,(\Pi^0 f)_K$, C_K is given by

$$C_K = -\frac{|K|}{\sum_{\bar e\in\partial K} |\bar e|^2}\,(\Pi^0 f)_K \tag{2.42}$$

and (2.36) follows from (2.41).
We prove now *(ii)*. Starting from (2.35), we have (cf (2.1) for the definition of $p_{K,e}$)

$$p_{h|K}\,(x) = \sum_{e\in\partial K} \frac{p_{K,e}(x)}{|e|}\left[v_{K,e}\nu_{K,e} + w_{K,e}\,\tau_{K,e}\right] \tag{2.43}$$

with $v_{K,e}$, $w_{K,e}$ given by (2.36). Since $\nabla\, u_h = \dfrac{\partial u_h}{\partial \nu_{K,e}}\,\nu_{K,e} + \dfrac{\partial u_h}{\partial \tau_{K,e}}\,\tau_{K,e}$ we get

$$p_{h|K}(x) = \nabla\, u_{h|K} - \frac{|K|\,(\Pi^0 f)_K}{\sum_{\bar e\in\partial K}|\bar e|^2} \sum_{e\in\partial K} p_{K,e}(x)\,|e|\,\nu_{K,e} \tag{2.44}$$

In each triangle K we may write the affine function $p_{K,e}\,(x)$ as

$$p_{K,e}(x) = \frac{1}{3} + \frac{|e|}{|K|}\,(\nu_{K,e}\cdot\overrightarrow{x_K x}) \tag{2.45}$$

Substituting (2.45) into (2.44) yields

$$p_{h|K}(x) = \nabla\, u_{h|K} - \frac{(\Pi^0 f)_K}{\sum_{\bar e}|\bar e|^2} \sum_{e\in\partial K} |e|^2\,(\nu_{K,e}\cdot\overrightarrow{x_K x})\,\nu_{K,e} \tag{2.46}$$

Thus, the gradient of $p_{h|K}$ is

$$\nabla\, p_{h|K} = -\frac{(\Pi^0 f)_K}{\sum_{\bar e}|\bar e|^2} \sum_{e\in\partial K} |e|^2\,\nu_{K,e}\otimes\nu_{K,e} \tag{2.47}$$

wich concludes *(ii)*.
Finally, we prove *(i)*. Suppose given $v_h(x) = \sum_{a\in A_i} v_a\,p_a\,(x)$ a function in $P^1_{nc,0}$. The restriction of v_h to any triangle K is $v_{h|K}\,(x) = \sum_{e\in\partial K} v_{K,e}\,p_{K,e}\,(x)$. We have the identity

$$0 = \sum_{a\in A_i}\int_a v_a\,[p_h\cdot\nu_a] = -\sum_K\sum_{e\in\partial K}\int_e v_{K,e}\,(p_h\cdot\nu_{K,e}) \tag{2.48}$$

For any $x\in e$, we have $v_{K,e} = v_{K,e}\,p_{K,e}\,(x) = v_h\,(x) - v_{K,e'}\,p_{K,e'}\,(x) - v_{K,e''}\,p_{K,e''}\,(x)$. Replacing $v_{K,e}$ by this value in (2.48), we get

$$0 = \underbrace{-\sum_K\sum_{e\in\partial K}\int_e v_h\,(p_h\cdot\nu_{K,e})}_{(I)} + \underbrace{\sum_K\sum_{e\in\partial K}\int_e [v_{K,e'}\,p_{K,e'}(x) + v_{K,e''}\,p_{K,e''}(x)]\,(p_h\cdot\nu_{K,e})}_{(II)} \tag{2.49}$$

Replacing $p_h(x)$ by its value (2.33) and taking in account $\int_e p_{K,e'} = \int_e p_{K,e''} = 0$ the second term is

$$(II) = \sum_K \sum_{e\in\partial K} \int_e v_{K,e'}\, p_{K,e'}\,(x)\,[\nu_{K,e}\cdot\nabla\, p_{h|K}\cdot\overrightarrow{x_K x}] + v_{K,e''}\, p_{K,e''}\,(x)\,[\nu_{K,e}\cdot\nabla\, p_{h|K}\cdot\overrightarrow{x_K x}] \tag{2.50}$$

Since $p_{K,e'}$, $p_{K,e''}$ are affine on e and $p_{K,e'}\,(m_e) = p_{K,e''}\,(m_e) = 0$, Simpson's quadrature rule yields

$$\begin{aligned}(II) \;=\; & \sum_K \sum_{e\in\partial K} \frac{|e|}{6} \Big\{ (v_{K,e''} - v_{K,e'}) \Big[\nu_{K,e}\cdot\nabla\, p_{h|K}\cdot\overrightarrow{x_K\, S'}\Big] \\ & - \;(v_{K,e''} - v_{K,e'}) \Big[\nu_{K,e}\cdot\nabla\, p_{h|K}\cdot\overrightarrow{x_K\, S''}\Big]\Big\}\end{aligned}$$

The identity $v_{K,e''} - v_{K,e'} = -\dfrac{|e|}{2}\,\dfrac{\partial v_h}{\partial \tau_{K,e}}$, allows to write

$$(II) = \sum_K \sum_{e\in\partial K} \frac{|e|^3}{12}\,(\nu_{K,e}\cdot\nabla\, p_{h|K}\cdot\tau_{K,e})\,\frac{\partial v_h}{\partial \tau_{K,e}} \tag{2.51}$$

We write now the local decomposition of $\nabla\, p_{h|K}$ in the basis $B_{K,e}$ of 2×2 matrices

$$B_{K,e} = \{\nu_{K,e}\otimes\nu_{K,e}\,,\,\nu_{K,e}\otimes\tau_{K,e}\,,\,\tau_{K,e}\otimes\nu_{K,e}\,,\,\tau_{K,e}\otimes\tau_{K,e}\}$$

We have

$$\nu_{K,e'} = -\cos\theta''\,\nu_{K,e} + \sin\theta''\,\tau_{K,e} \quad ; \quad \nu_{K,e''} = -\cos\theta'\,\nu_{K,e} - \sin\theta'\,\tau_{K,e} \tag{2.52}$$

Replacing $\nu_{K,e'}\otimes\nu_{K,e'}$ and $\nu_{K,e''}\otimes\nu_{K,e''}$ by their values in function of the elements of $B_{K,e}$, we obtain

$$\begin{aligned}\nabla\, p_{h|K} \;=\; & -\frac{(\Pi^0 f)_K}{\sum_{\bar e}|\bar e|^2}\Big[\Big\{|e|^2 + |e'|^2\,(\cos\theta'')^2 + |e''|^2\,(\cos\theta')^2\Big\}\;\nu_{K,e}\otimes\nu_{K,e} \\ & +\;\Big\{|e'|^2\,(\sin\theta'')^2 + |e''|^2\,(\sin\theta')^2\Big\}\,\tau_{K,e}\otimes\tau_{K,e} \\ & +\;\Big\{-\frac{2|K|}{|e|^2}\;\big[|e'|^2 - |e''|^2\big]\Big\}\;[\nu_{K,e}\otimes\tau_{K,e} + \tau_{K,e}\otimes\nu_{K,e}]\Big]\end{aligned}$$

The component of $\nabla\, p_{h|K}$ onto $\nu_{K,e}\otimes\tau_{K,e}$ is

$$\nu_{K,e}\cdot\nabla\, p_{h|K}\cdot\tau_{K,e} = \frac{2|K|\,(\Pi^0 f)_K}{\big[\sum_{\bar e}|\bar e|^2\big]\,|e|^2}\,(|e'|^2 - |e''|^2)$$

and we deduce from (2.51).

$$(II) = \sum_K |K|\,(\Pi^0 f)_K\;\left[\nabla\, v_h\cdot\left\{\frac{1}{6\sum_{\bar e}|\bar e|^2}\;\sum_{e\in\partial K}(|e'|^2 - |e''|^2)\,|e|\,\tau_{K,e}\right\}\right] \tag{2.53}$$

Defining for any K the vector A_K by

$$A_K = \frac{1}{6\sum_{\bar e}|\bar e|^2}\,\sum_{e\in\partial K}(|e'|^2 - |e''|^2)\,|e|\,\tau_{K,e} \tag{2.54}$$

we obtain

$$(II) = \sum_K |K|\,(\Pi^0 f)_K\,(\nabla v_h \cdot A_K) = \sum_K \int_K f(\nabla v_h \cdot A_K) \tag{2.55}$$

Summarizing (I) and (II) we deduce

$$\begin{aligned} 0 &= -\sum_K \sum_{e\in\partial K} \int_e v_h\,(p_h \cdot \nu_{K,e}) + \sum_K |K|(\Pi^0 f)_K(\nabla v_h \cdot A_K) \\ &= -\sum_K (p_h\,,\,\nabla v_h)_{0,K} - \sum_K (\text{div}\ p_h\,,\,v_h)_{0,K} + \sum_K |K|(\Pi^0 f)_K(\nabla v_h \cdot A_K) \\ &= -\sum_K (\nabla u_h\,,\,\nabla v_h)_{0,K} + \sum_K (f\,,\,(\Pi^0 v_h))_{0,K} + \sum_K (f\,,\nabla v_h \cdot A_K)_{0,K} \end{aligned}$$

Which is (2.31). Note that the vectors A_K verify an estimate

$$\sup_{K\in\mathcal{T}_h} |A_K| \le Ch \tag{2.56}$$

Where C is independant of h.

Let us check now the equivalence between the FVbox scheme (2.22) and the scheme (2.31, 2.35). The scheme (2.22) has an unique solution by Lemma 2.1, which is also solution of scheme (2.31, 2.35). Since this lattest scheme has clearly at most one solution, it has exactly one solution, which is the solution of (2.22). ∎

Note that it results from (2.31) that the global linear system in the unknowns $(u_a)_{a\in A_i}$ is in fact symmetric definite positive. In order to deduce from the preceding results the well-posedness of discrete problem (2.22), we need the following discrete counterpart of the Poincaré inequality in the space $H^1_0 + P^1_{nc,0}$, [16]. Let us mention that the proof given in [10] is incomplete, since the inequality is proved there separately for $u \in H^1_0$, $u \in P^1_{nc,0}$ but not for $u \in H^1_0 + P^1_{nc,0}$. Moreover, the hypothesis made in [10] on the mesh is useless. We give a proof for completeness.

Lemma 2.3. *For $u \in H^1_0 + P^1_{nc,0}$, we have*

$$|u|_{0,\Omega} \le C\,|u|_{1,h}$$

where C is a constant depending only of Ω.

Proof. We have

$$|u|_{0,\Omega} = \sup_{g\in L^2,\,g\neq 0} \frac{(u,g)_{0,\Omega}}{|g|_{0,\Omega}} \tag{2.57}$$

For any $g \in L^2(\Omega)$, there exists $p \in H^1(\Omega)^2$ such that

$$\text{div}\ p = g \quad , \quad ||p||_{1,\Omega} \le C(\Omega)\,|g|_{0,\Omega} \tag{2.58}$$

$$(u,g)_{0,\Omega} = (\text{div}\ p, u)_{0,\Omega} = \underbrace{-\sum_K (\nabla u, p)_{0,\Omega}}_{(I)} + \underbrace{\sum_K \int_{\partial K} (p\cdot\nu_K)\,u}_{(II)} \tag{2.59}$$

A classical calculation [6] yields

$$\begin{aligned}(II) &= -\sum_{a\in A_i}\int_a (p\cdot\nu_a)\,[u] + \sum_{a\in A_b}\int_a (p\cdot\nu_a)\,u \\ &= -\sum_{a\in A_i}\int_a (p\cdot\nu_a - \overline{p\cdot\nu_a})\,[u] + \sum_{a\in A_b}\int_a (p\cdot\nu_a - \overline{p\cdot\nu_a})\,u \\ &= \sum_K \sum_{e\in\partial K}\int_e (p\cdot\nu_{K,e} - \overline{p\cdot\nu_{K,e}})\,u\end{aligned}$$

which gives by Lemma 3 of [12] the estimate

$$|(II)| \le \sum_K C\,h\,(|p|_{1,K}\,|u|_{1,K}) \le C\,h\,|p|_{1,\Omega}\,|u|_{1,h}$$

where C is independant of h. Finally

$$|(u,g)_{0,\Omega}| \le |p|_{0,\Omega}\,|u|_{1,h} + C\,h\,|p|_{1,\Omega}\,|u|_{1,h} \le C\,(\Omega)\,(C\,h+1)\;|g|_{0,\Omega}\,|u|_{1,h}$$

and the result follows by dividing by $|g|_{0,\Omega}$ on each side. ■

Proposition 2.4. *The unique solution* $(u_h\,,\,p_h)\in M_{1,h}\times X_{1,h}$ *of problem* (2.22) *verifies*

$$\|u_h\|_{1,h} + \|p_h\|_{1,h} \le C|f|_{0,\Omega} \tag{2.61}$$

Proof. Taking $v_h = u_h$ in (2.31) and using the fact that the vectors A_K are in $O(h)$, we deduce

$$\begin{aligned}|u_h|^2_{1,h} &= \sum_K (f\,,\,(\Pi^0\,u_h) + \nabla\,u_h\cdot A_K)_{0,K} &(2.62)\\ &\le |f|_{0,\Omega}\,(|u_h|_{0,\Omega} + C\,h\,|u_h|_{1,h}) &(2.63)\end{aligned}$$

Using Lemma 2.3, we obtain, $\|u_h\|_{1,h}\le C\,|f|_{0,\Omega}$. From the local representation formula (2.46) for p_h , we deduce

$$|p_h|_{0,\Omega} \le |u_h|_{1,h} + C\,h\,|f|_{0,\Omega} \le C'\,|f|_{0,\Omega} \tag{2.64}$$

In addition, $|p_h|_{1,h}\le C|f|_{0,\Omega}$ results from (2.34) ■

Recall that well-posedness of discrete problem (2.22) is equivalent to the two following conditions *(i)*,*(ii)* [3, 5, 7, 22, 11], applied to the bilinear form B_h defined on $(M_{1,h}\times X_{1,h})\times(M_{2,h}\times X_{2,h})$ by

$$B_h\,[(u_h\,,\,p_h)\;;\;(v_h\,,\,q_h)] = (p_h\,,\,q_h)_{0,\Omega} + \sum_K(\text{div}\;p_h\,,\,v_h)_{0,K} - \sum_K(q_h\,,\,\nabla\,u_h)_{0,K} \tag{2.65}$$

(i) $\exists\alpha>0$ such that

$$\sup_{\|q_h,v_h\|_{2,h}\le 1} B_h\,[(u_h\,,\,p_h)\,;\,(v_h\,,\,q_h)] \ge \alpha\,\|(u_h\,,\,p_h)\|_{1,h} \qquad \forall(u_h,p_h)\in M_{1,h}\times X_{1,h}$$

(ii) For any $(v_h\,,\,q_h)\in M_{2,h}\times X_{2,h}$

$$\forall\,(u_h\,,\,p_h)\in M_{1,h}\times X_{1,h}\quad,\quad B_h\,[(u_h\,,\,p_h)\,;\,(v_h\,,\,q_h)] = 0 \Longrightarrow (v_h\,,\,q_h) = (0,0)$$

In addition, well-posedness of scheme (2.22) is equivalent to the one of the dual scheme: find $(v_h\,,\,q_h) \in M_{2,h} \times X_{2,h}$ such that for any $(u_h\,,\,p_h) \in M_{1,h} \times X_{1,h}$

$$B_h\left[(u_h\,,\,p_h)\;;\;(v_h\,,\,q_h)\right] = -(f\,,\,u_h) \tag{2.68}$$

The latter scheme can be rewritten as, [11]:
find $(v_h\,,\,q_h) \in P^0 \times \Big[(P^0)^2 + Vect\{Q_{K,e}, K \in \mathcal{T}_h, e \in \partial K\}\Big]$ such that

$$\begin{cases} -\displaystyle\sum_K (\nabla\, u_h\,,\,q_h)_{0,K} = -(f, u_h)_{0,\Omega} & \forall u_h \in P^1_{nc,0} \\ (p_h\,,\,q_h)_{0,\Omega} + (\mathrm{div}\, p_h, v_h)_{0,\Omega} = 0 & \forall p_h \in (P^1_{nc})^2 \end{cases} \tag{2.69}$$

The standard error estimates for the two schemes are

Proposition 2.5.
(i) The solution $(u_h\,,\,p_h)$ *of scheme* (2.22) *verifies the error estimate*

$$\|u - u_h\|_{1,h} + \|p - p_h\|_{1,h} \le C\,h\left[|u|_{2,\Omega} + |u|_{3,\Omega}\right] \tag{2.70}$$

(ii) The solution $(v_h\,,\,q_h)$ *of scheme* (2.69) *verifies the error estimate*

$$|u - v_h|_{0,\Omega} + |p - q_h|_{0,\Omega} \le C\,h\,|u|_{2,\Omega} \tag{2.71}$$

Proof. The proof follows the same lines than the one in [11]. We have

$$\begin{aligned} \|u - u_h\|_{1,h} + \|p - p_h\|_{1,h} \;\le\; C \Big\{ & \inf_{(\tilde u_h, \tilde p_h) \in M_{1,h} \times X_{1,h}} \Big[\|u - \tilde u_h\|_{1,h} + \|p - \tilde p_h\|_{1,h}\Big] \\ & + \sup_{(\tilde v_h, \tilde q_h) \in M_{2,h} \times X_{2,h}} \frac{|B_h\left[(u,p)\;;\;(\tilde v_h\,,\,\tilde q_h)\right] + (f\,,\,\tilde v_h)_{0,\Omega}|}{\|(\tilde v_h\,,\,\tilde q_h)\|_{H_{2,h}}} \Big\} \end{aligned}$$

It is straightfoward to check, that, as in [11], the consistancy error vanishes. Therefore, (i) results simply from the two standard interpolation estimates in spaces $M_{1,h}$, $X_{1,h}$

$$\inf_{\tilde u_h \in M_{1,h}} \|u - \tilde u_h\|_{1,h} \le C\,h\,|u|_{2,\Omega} \quad , \quad \inf_{\tilde p_h \in X_{1,h}} \|p - \tilde p_h\|_{1,h} \le C\,h\,|p|_{2,\Omega} \tag{2.72}$$

For (ii) we have

$$\begin{aligned} |u - v_h|_{0,\Omega} + |p - q_h|_{0,\Omega} \;\le\; C \Big\{ & \inf_{(\tilde v_h, \tilde q_h) \in M_{2,h} \times X_{2,h}} \Big[|u - \tilde v_h|_{0,\Omega} + |p - \tilde q_h|_{0,\Omega}\Big] \\ & + \sup_{(\tilde u_h, \tilde p_h) \in M_{1,h} \times X_{1,h}} \frac{|B_h\left[(\tilde u_h\,,\,\tilde p_h)\;;\;(u,p)\right] + (f\,,\,\tilde u_h)_{0,\Omega}|}{\|(\tilde u_h\,,\,\tilde p_h)\|_{H_{1,h}}} \Big\} \end{aligned}$$

The consistancy error is written as

$$\begin{aligned} B_h\left[(\tilde u_h\,,\,\tilde p_h)\;;\;(u,p)\right] + (f\,,\,\tilde u_h)_{0,\Omega} &= (\tilde p_h\,,\,p)_{0,\Omega} + \sum_K (\mathrm{div}\;\tilde p_h\,,\,u)_{0,K} - \sum_K (\nabla\,\tilde u_h\,,\,p)_{0,K} + (f\,,\,\tilde u_h)_{0,\Omega} \\ &= \underbrace{\sum_K \int_{\partial K} (\tilde p_h \cdot \nu_K)\, u}_{(I)} - \underbrace{\sum_K \int_{\partial K} (p \cdot \nu_K)\, \tilde u_h}_{(II)} \end{aligned}$$

A classical argument gives :

$$|(I)| \le C\,h\,|\tilde p_h|_{1,h}\,|f|_{0,\Omega} \quad ; \quad |(II)| \le C\,h\,|f|_{0,\Omega}\,|\tilde u_h|_{1,h} \tag{2.73}$$

The result follows from the two standard interpolation estimates

$$\inf_{\tilde{v}_h \in M_{2,h}} |u - \tilde{v}_h|_{0,\Omega} \leq C\, h\, |u|_{1,\Omega} \quad ; \quad \inf_{\tilde{q}_h \in X_{2,h}} |p - \tilde{q}_h|_{0,\Omega} \leq C\, h\, |p|_{1,\Omega} \tag{2.74}$$

■

Another possibility to derive error estimate (2.70) is to use the reduced scheme (2.31) in u_h. We can prove that

$$\|u - u_h\|_{1,h} \leq C\, h\, |u|_{2,\Omega}\,.$$

Error estimate (2.70) for p_h is derived in a second step from the representation formula (2.33).
Finally, we have the following second order error estimate in the L^2 norm whose proof follows the same lines than the one in [11].

Proposition 2.6. *The solution $u_h \in P^1_{nc,0}$ verifies*

$$|u - u_h|_{0,\Omega} \leq C\, h^2\, (|u|_{2,\Omega} + |u|_{3,\Omega}) \tag{2.75}$$

III. THE CASE $(\boldsymbol{BDM^1} + \boldsymbol{\nabla^\perp B_K^{NC}}) \times \boldsymbol{P^2_{NC,0}}$

A. Discrete equations

In this section, we describe another FVbox scheme still having form (1.2). This scheme is closely connected to the non-conforming piecewise quadratic method of Fortin and Soulie [15], and to its interpretation as a mixed method given by Farhloul and Fortin in [14]. In fact, we prove that our scheme is nothing but the hybridization of this method. Let us firstly recall the two spaces usefull in the sequel. The first space is the space $M_{1,h} = P^2_{nc,0}$ of scalar quadratic functions continuous at the two Gaussian nodes on each interface $a \in \mathcal{A}_i$, vanishing at the Gaussian nodes of each boundary edge $a \in \mathcal{A}_b$. An important feature of this space is that the values of $u_{h|K} \in P^2_{nc,0}$ at the six gaussian points on ∂K are not an unisolvant set of linear forms. Indeed, there is a non-trivial function vanishing at these six points, which is the *non-conforming quadratic bubble function* given by

$$b_K^{nc} = 2 - 3\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \tag{3.1}$$

It is proved in [15] that $M_{1,h} = \hat{M}_{1,h} \oplus \tilde{M}_{1,h}$ where $\hat{M}_{1,h}$ is the quadratic conforming space $P^2_{c,0}$ with homogeneous boundary Dirichlet conditions and $\tilde{M}_{1,h}$ is the space $\tilde{M}_{1,h} = \{w_h \;;\; w_{h|K} = \alpha_K\, b_K^{nc} \;,\; \alpha_K \in \mathbb{R} \quad \forall K \in \mathcal{T}_k\}$. Thus, the dimension of $M_{1,h}$ is

$$\dim\ M_{1,h} = \dim \hat{M}_{1,h} + \dim \tilde{M}_{1,h} = NA_i + NP_i + NE \tag{3.2}$$

Thanks to the Euler relation $NE - NA + NP = 1$, we deduce that $\dim\ M_{1,h} = 2NA_i + 1$.

In addition, keeping notation (1.4) for the norms onto the space $H^1_0 + M_{1,h}$, we have the following result, whose proof is analogous to the one of Lemma 2.3

Lemma 3.1. *The semi-norm $|\ \ |_{1,h}$ is a norm on $M_{1,h}$, equivalent to $\|u\|_{1,h}$*

For the space associated with the flux p_h, we use the space introduced in [14], $X_{1,h} = \hat{X}_{1,h} + \tilde{X}_{1,h}$ where $\hat{X}_{1,h} = BDM^1$ is the space of Brezzi-Douglas-Marini [8] of lowest order defined by

$$BDM^1 = \Big\{ p_h \in H_{\rm div}\,(\Omega)\ ;\ p_h \in (P^1\,(K))^2 \quad \forall\, K \in \mathcal{T}_h \Big\} \tag{3.3}$$

and $\tilde{X}_{1,h}$ is defined by

$$\tilde{X}_{1,h} = \{ p_{|K} = \alpha_K \nabla^{\perp} b_K^{nc}\,,\ \alpha_K \in \mathbb{R} \quad \forall\, K \in \mathcal{T}_h \} \tag{3.4}$$

The space $\hat{X}_{1,h} \cap \tilde{X}_{1,h}$ reduces to the one dimensional space generated by the function $\sum\limits_{K\in\mathcal{T}_h} \nabla^{\perp} b_K^{nc}$. Consequently, we have $\dim X_{1,h} = 2NA + NE - 1$. Recall, [14], that $X_{1,h}$ coincides with the space of vectorial functions p_h, affine in each $K \in \mathcal{T}_h$ and verifying the two weakened div-conformity properties:

$$\left\{ \begin{array}{l} (i) \quad \int_e (p_{h|K_1} \cdot \nu_{e,K_1} + p_{h|K_2} \cdot \nu_{e,K_2}) = 0 \quad \text{for any } e = \partial K_1 \cap \partial K_2 \\ (ii) \text{ For any internal vertex } M, \quad \sum_K \int_{\partial K} (p_h \cdot \nu) \psi_M = 0 \text{ where } \psi_M \\ \text{is the standard } P^1\text{-Lagrange function corresponding to } M. \end{array} \right. \tag{3.5}$$

We have now to define the test spaces $M_{2,h}$, $X_{2,h}$ as discontinuous Galerkin spaces. In order to keep the relation

$$\dim M_{2,h} + \dim X_{2,h} = \dim M_{1,h} + \dim X_{1,h} = 7NE \tag{3.6}$$

we take $M_{2,h} = \{ v_h \in L^2(\Omega)\ ;\ v_{h|K} \in P^0(K) \quad \forall\, K \in \mathcal{T}_h \}$ (dim $M_{2,h} = NE$) and $X_{2,h} = \{ q_h \in (L^2(\Omega))^2\ ;\ q_{h|K} \in (P^1\,(K))^2 \quad \forall\, K \in \mathcal{T}_h \}$ (dim $X_{2,h} = 6NE$). The discrete system has still form (1.2): find $(p_h\,,\,u_h) \in X_{1,h} \times M_{1,h}$ such that

$$\left\{ \begin{array}{l} \sum\limits_{K\in\mathcal{T}_h} (\operatorname{div} p_h + f, v_h)_{0,K} = 0 \quad \forall v_h \in M_{2,h} \\ \sum\limits_{K\in\mathcal{T}_h} (p_h - \nabla u_h, q_h)_{0,K} = 0 \quad \forall q_h \in X_{2,h} \end{array} \right. \tag{3.7}$$

The following proposition states that this method is nothing but the scheme of Fortin and Soulie [15]

Proposition 3.2. *Problem* (3.7) *has an unique solution* $(u_h\,,\,p_h) \in M_{1,h} \times X_{1,h}$ *given by*
(i) $u_h \in M_{1,h}$ *is the solution of the standard variational problem:*

$$\sum_K (\nabla u_h\,,\,\nabla w_h)_{0,K} = (\Pi^0\,f\,,\,w_h)_{0,\Omega} \quad \forall\, w_h \in M_{1,h} \tag{3.8}$$

(ii) p_h *is given by*

$$p_{h|K} = \nabla\, u_{h|K} \tag{3.9}$$

Proof. We prove that $(u_h, p_h) \in M_{1,h} \times X_{1,h}$ is solution of (3.7) if and only if it is solution of (3.8), (3.9). Suppose given $(u_h, p_h) \in M_{1,h} \times X_{1,h}$ a solution of (3.7). Clearly, we have $p_h = \nabla\, u_h$ (take in $(3.7)_b$ $q_h = p_h - \nabla\, u_h$). Let $w_h \in M_{1,h}$, Green's formula yields

$$\sum_K (\nabla\, u_h\,,\,\nabla\, w_h)_{0,K} = -\sum_K \int_K \operatorname{div}\ p_h\, w_h + \sum_K \int_{\partial K} (p_h \cdot \nu_K)\ w_h \tag{3.10}$$

Let us write $p_h = p_{1,h} + p_{2,h}$, with $p_{1,h} \in \hat{X}_{1,h}$, $p_{2,h} = \sum_{K \in \mathcal{T}_h} \alpha_K \nabla^\perp b_K^{nc}$. Since $\hat{X}_{1,h} \subset H_{\rm div}$, the second term in the right-hand side of (3.10) may be rewritten

$$\begin{aligned} \sum_K \int_{\partial K} (p_h \cdot \nu_K)\, w_h \; &= \; \sum_K \Big[\int_{\partial K} (p_{1,h} \cdot \nu_K)\, w_h \; + \alpha_K \int_{\partial K} (\nabla^\perp b_K^{nc} \cdot \nu_K)\, w_h \,\Big] \\ &= \; \sum_{a \in A_b} \int_a (p_{1,h} \cdot \nu_a)\, w_h \; - \sum_{a \in A_i} \int_a (p_{1,h} \cdot \nu_a)\, [w_h] \\ &+ \; \sum_K \alpha_K \int_{\partial K} \frac{\partial b_K^{nc}}{\partial \tau_K}\, w_h \end{aligned}$$

For any $w_h \in P^2(K)$, we have $\int_{\partial K} \frac{\partial b_K^{nc}}{\partial \tau_{K,e}}\, w_h \, d\sigma = 0$. Therefore, the third term vanishes. In addition, since $[w_h]$ (resp. w_h) vanishes at Gauss nodes of each internal (resp. boundary) edge, the second and first term vanish and (3.8) results from $(3.7)_a$. Consequently, any $(u_h, p_h) \in M_{1,h} \times X_{1,h}$ solution of problem (3.7) is solution of problem (3.8, 3.9), which admits an unique solution in $M_{1,h} \times (P^1_{t.d.})^2$ where $(P^1_{t.d.})^2$ is the Galerkin-discontinuous space of affine functions p_h in each triangle. This gives uniqueness of the solution of (3.7). The existence follows from (3.6) and the linearity of (3.7). In particular, we have proved that the function $\nabla u_{h|K}$ is in fact in $X_{1,h}$. ∎

Proposition 3.3. *The following error estimates hold*

$$\|u - u_h\|_{1,h} \le C\, h^2\, (|u|_{3,\Omega} + |\Delta u|_{1,\Omega}) \quad ; \quad |u - u_h|_{0,\Omega} \le C\, h^2 (|u|_{3,\Omega} + |\Delta u|_{1,\Omega}) \tag{3.11}$$

$$|p - p_h|_{0,\Omega} \le C\, h^2\, |\Delta u|_{0,\Omega} \quad ; \quad |p - p_h|_{h,\rm div} \le C\, h\, |\Delta u|_{1,\Omega} \tag{3.12}$$

Proof. Results from [15] and from $p_h = \nabla u_h$. ∎

B. Comparaison with the mixed method of Farhloul and Fortin

In [14], addressing the paper by Hiptmair [18], Farhloul and Fortin have introduced the following mixed method: find $(u'_h, p'_h) \in M_{2,h} \times X_{1,h}$ such that

$$\left\{ \begin{array}{l} \displaystyle\sum_{K \in \mathcal{T}_h} ({\rm div}\; p'_h + f \,,\, v_h)_{0,K} = 0 \qquad \forall v_h \in M_{2,h} \\ \displaystyle\sum_{K \in \mathcal{T}_h} \{ (p'_h \,,\, q_h)_{0,K} + ({\rm div}\; q_h \,,\, u'_h)_{0,K} \} = 0 \qquad \forall q_h \in X_{1,h} \end{array} \right. \tag{3.13}$$

This problem has an unique solution $(u'_h\,,\, p'_h) \in M_{2,h} \times X_{1,h}$ satisfying $\|u'_h\|_{1,h} + \|p'_h\|_{{\rm div},h}\| \le C |f|_{0,\Omega}$. The hybrid form of (3.13) is: find $(p_h\,,\, u_h\,,\, \lambda_h) \in X_{2,h} \times M_{2,h} \times \Lambda_h$ such that for any $(q_h\,,\, v_h\,,\, \mu_h) \in X_{2,h} \times M_{2,h} \times \Lambda_h$

$$\left\{ \begin{array}{l} \displaystyle\sum_{K \in \mathcal{T}_h} \Big\{ (p_h\,,\, q_h)_{0,K} + \int_K ({\rm div}\; q_h)\, u_h \, dx - \int_{\partial K} (q_h \cdot \nu_K)\, \lambda_h \, d\sigma \Big\} = 0 \\ \displaystyle\sum_{K \in \mathcal{T}_h} \int_K ({\rm div}\; p_h)\, v_h \; + (f\,,\, v_h)_{0,\Omega} = 0 \\ \displaystyle\sum_{K \in \mathcal{T}_h} \int_{\partial K} (p_h \cdot \nu_K)\; \mu_h \; = 0 \end{array} \right. \tag{3.14}$$

The space Λ_h of Lagrange multipliers is defined by $\Lambda_h = \Lambda_{1,h} \oplus \Lambda_{2,h}$ with

$$\Lambda_{1,h} = \Big\{ \mu_h \in L^2(\Gamma_h) \ ; \ \mu_{h|_e} \in P^0\,(e) \quad \forall\, e \in \Gamma_h^0 \ , \ \mu_{h|_e} = 0 \quad \forall\, e \in \Gamma_h^{\partial} \Big\} \tag{3.15}$$

$$\Lambda_{2,h} = \Big\{ \mu_h = \psi|_{\Gamma_h} \ ; \ \psi \in C^0(\overline{\Omega})\,,\, \psi|_K \in P^1(K)\,,\, \forall\, K \in \mathcal{T}_h \,,\, \psi|_\Gamma = 0 \Big\} \tag{3.16}$$

Γ_h denotes the set of edges of the element of $\mathcal{T}_h$, $\Gamma_h^{\partial} = \{e \in \Gamma_h \ ; \ e \subset \partial\Omega\}$, $\Gamma_h^0 = \Gamma_h \setminus \Gamma_h^{\partial}$. Let us recall that $u_h \in P^2_{nc,0}$ writes $u_h = u_h^c + u_h^{nc}$ where $u_h^c \in P^2_{c,0}$, $u_h^{nc} \in \tilde{M}_{1,h}$ denote respectively the conforming and non-conforming part of u_h .
The link between the box scheme (3.7) and the hybrid formulation (3.14) of the mixed method (3.13) is given by the

Proposition 3.4. *Let* $(u_h\,,\,p_h) \in M_{1,h} \times X_{1,h}$ *be the solution of problem* (P_h)*, let* $\lambda_h \in \Lambda_h = \Lambda_{1,h} \oplus \Lambda_{2,h}$ *be defined by* $\lambda_h = \lambda_{1,h} + \lambda_{2,h}$ *where* $\lambda_h^1 \in \Lambda_{1,h}$ *is defined for* $a = [S', S''] \in \mathcal{A}_i$ *by*
(i) $\lambda^1_{h|a} = \frac{1}{3}[2u_h^c\,(x_a) - (u_h^c\,(S') + u_h^c\,(S''))] \qquad \forall\, a \in \mathcal{A}_i$
where u_h^c *denotes the conforming part of* u_h*.*
(ii) $\lambda_h^2 \in \Lambda_{2,h}$ *is the affine continuous function defined by the values of* u_h^c *at the vertices of the mesh. Then*
(a) $(p_h\,,\,\Pi^0\,u_h)$ *is the solution of mixed scheme* (3.13)*.*
(b) $(p_h\,,\,\Pi^0\,u_h\,,\,\lambda_h)$ *is the solution of* (3.14)*.*

Proof. *(a)* Let $(u_h\,,\,p_h) \in M_{1,h} \times X_{1,h}$ be the solution of the FVbox scheme (3.7). Since $(3.7)_a$ and $(3.13)_a$ are identical, we just have to check that $(p_h\,,\,\Pi^0\,u_h)$ is solution of equation $(3.13)_b$. For q_h in $X_{1,h}$, since $\operatorname{div} q_h \in P^0$, we have

$$(p_h\,,\,q_h) = \sum_{K\in\mathcal{T}_h} (\nabla\, u_h\,,\,q_h)_{0,K} = -\sum_{K\in\mathcal{T}_h} (\Pi^0 u_h\,,\,\operatorname{div}\,q_h)_{0,K} + \sum_{K\in\mathcal{T}_h} \int_{\partial K} u_h\,(q_h \cdot \nu_K) \tag{3.17}$$

It results from the proof of Prop. 3.2, that the second sum in (3.17) vanishes, which gives $(3.13)_b$.
(b) By unicity of the solution of (3.14), it is sufficient to check that $(\Pi^0 u_h, p_h, \lambda_h)$ defined in Prop 3.4 is solution of (3.14). For $q_h \in X_{2,h}$, we have still (3.17).
Defining $\lambda = (\lambda_1, \lambda_2) \in \Lambda_{1,h} \times \Lambda_{2,h}$ by *(i), (ii)*, we deduce easily from Simpson's quadrature formula, that for any $e \in \partial K$,

$$\int_e (q_h \cdot \nu_{K,e}) u_h^c = \int_e (q_h \cdot \nu_{K,e}) \lambda \tag{3.18}$$

Therefore, $(\Pi^0 u_h, p_h, \lambda_h) \in M_{2,h} \times X_{2,h} \times \Lambda_h$ is solution of (3.14). ∎

IV. THE CASE $(\boldsymbol{RT^1} + \boldsymbol{\nabla}^{\perp}\boldsymbol{B_K^{NC}}) \times \boldsymbol{P^2_{NC,0}}$

A. Discrete equations

In this third scheme, we introduce a new space for the approximation of the vectorial unknow p_h which is $X_{1,h} = RT^1 + \tilde{X}_{1,h}$ where RT^1 is the standard Raviart-Thomas space of order 1 (see [23]) defined by

$$RT^1 = \Big\{ p_h \in H_{\mathrm{div}}(\Omega) \ ; \ p_{h|K} \in RT_1\,(K) \quad \forall\, K \in \mathcal{T}_h \Big\} \tag{4.1}$$

where for each $K \in \mathcal{T}_h$, $RT^1(K) = P_1(K)^2 + P_1(K)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. The space $\tilde{X}_{1,h}$ is introduced in Sect. III.A. Again, $\dim RT^1 \cap \tilde{X}_{1,h} = 1$ and $RT^1 \cap \tilde{X}_{1,h} = Vect\left(\sum_{K\in\mathcal{T}_h} \nabla^{\perp} b_K^{nc}\right)$. Using $\dim RT^1(K) = 8$, and (1.8), we deduce $\dim X_{1,h} = 3NE + 2NA - 1$. In addition, using the same method than in [15], we check that $X_{1,h}$ has the following characterization: $p_h \in X_{1,h}$ if and only if $p_{h|K} \in RT^1(K)$ and p_h verifies the two weakened div-conformity conditions (3.5). We still use $M_{1,h} = P^2_{nc,0}$ for the scalar unknow u_h whose dimension is $2NA_i + 1$. Consequently, $\dim X_{1,h} + \dim M_{1,h} = 3NE + 2(NA + NA_i) = 9NE$. This suggests to take for the test functions the following Galerkin discontinuous spaces

$$X_{2,h} = \left\{ q_h \in (L^2(\Omega))^2 \ ; \ q_{h|_K} \in (P_1(K))^2, \quad \forall K \in \mathcal{T}_h \right\} \quad , \dim X_{2,h} = 6NE \qquad (4.2)$$

$$M_{2,h} = \left\{ v_h \in L^2(\Omega) \ ; \ v_{h|K} \in P_1(K) \quad , \forall K \in \mathcal{T}_h \right\} \quad , \ \dim M_{2,h} = 3NE \qquad (4.3)$$

We have $\dim X_{2,h} + \dim M_{2,h} = 9NE$. The FVbox scheme reads still: find $(u_h\, , p_h) \in M_{1,h} \times X_{1,h}$ such that

$$\left\{ \begin{array}{l} \displaystyle\sum_{K\in\mathcal{T}_h} (\operatorname{div} p_h + f, v_h)_{0,K} = 0 \quad \forall v_h \in M_{2,h} \\ \displaystyle\sum_{K\in\mathcal{T}_h} (p_h - \nabla u_h, q_h)_{0,K} = 0 \quad \forall q_h \in X_{2,h} \end{array} \right. \qquad (4.4)$$

Proposition 4.1. *Problem* (4.4) *has an unique solution* $(u_h\, , p_h)$ *in* $M_{1,h} \times X_{1,h}$, *given by*
(i) $u_h \in M_{1,h}$ *is solution of*

$$\sum_K (\nabla u_h\, , \nabla w_h)_{0,K} = ((\Pi^1 f)\, , w_h)_{0,\Omega} \qquad \forall w_h \in M_{1,h} \qquad (4.5)$$

where Π^1 *is the orthogonal projector onto the affine functions in each triangle* K.
(ii) p_h *is locally given by*

$$p_{h|K} = \nabla u_{h|K} - \frac{1}{3}\left\{ (\Pi^1 f)\, \overrightarrow{x_K\, x} - \Pi^1[(\Pi^1 f)\, \overrightarrow{x_K\, x}] \right\} \qquad (4.6)$$

Proof. Problem (4.4) is linear in (u_h, p_h) and the number of unknowns is equal to the number of equations. Therefore we just have to prove unicity of the solution of (4.4) which is given by unicity of problem(4.5-4.6) .
(i) Suppose $(u_h\, , p_h) \in M_{1,h} \times X_{1,h}$ satisfies (4.4). For any $w_h \in P^2_{nc,0}$, one has $\nabla w_h \in X_{2,h}$. Therefore, by $(4.4)_a$ and Green's formula

$$\sum_K (\nabla u_h\, , \nabla w_h)_{0,K} = -\sum_K (\operatorname{div}\, p_h\, , w_h)_{0,K} + \sum_K \int_{\partial K} (p_h \cdot \nu_K)\, w_h \qquad (4.7)$$

The decomposition of $p_{h|K} \in X_{1,h} = RT^1 + \nabla^{\perp} b^{nc}$ writes $p_{h|K} = \hat{p}_{h|K} + \alpha_K \nabla^{\perp} b^{nc}_K$. Therefore, the second part of the r.h.s. of (4.7) is $\displaystyle\sum_K \int_{\partial K} (\hat{p}_h \cdot \nu_K)\; w_h + \sum_K \alpha_K \int_{\partial K} \frac{\partial b^{nc}_K}{\partial \tau_K} w_h$.
As in the proof of Prop. 3.2, we deduce

$$\sum_K (\nabla u_h\, , \nabla w_h)_{0,K} = -\sum_K (\operatorname{div} p_h\, , w_h)_{0,K} \qquad (4.8)$$

Since $M_{2,h}$ is the Galerkin discontinuous P^1 space, we deduce from $(4.4)_a$ that $\operatorname{div} p_{h|K} = -\Pi^1 f$, which gives (4.5).
(ii) Consider now $p_{h|K} = \hat{p}_{h|K} + \alpha_K \nabla^\perp b_K^{nc}$. The local expression of $\hat{p}_{h|K} \in RT^1(K)$ is $\hat{p}_{h|K} = \overline{p}_{h|K} + [A_K \cdot \overrightarrow{x_K x}]\,\overrightarrow{x_K x}$, where $\overline{p}_h \in P^1(K)^2$ and A_K is a constant vector in K. The divergence of p_h reduces to

$$\operatorname{div}\, p_h = \operatorname{div}\, \overline{p}_h + 3\,(A_K \cdot \overrightarrow{x_K x}) \tag{4.9}$$

Since $\operatorname{div} p_h = -\Pi^1 f$, $A_K = -\frac{1}{3}\nabla(\Pi^1 f)$.
In addition, we have $(\Pi^1 f)_K = (\Pi^0 f)_K + \nabla(\Pi^1 f)_K \cdot \overrightarrow{x_K x}$, thus, the quadratic part of $p_{h|K}$ is $(A_K \cdot \overrightarrow{x_K x})\overrightarrow{x_K x} = -\frac{1}{3}[(\Pi^1 f - \Pi^0 f)\overrightarrow{x_K x}]$. The remaining part $\tilde{p}_{h|K} = \overline{p}_{h|K} + \alpha_K \nabla^\perp b_K^{nc}$ is linear and is determined, by (4.4),

$$\sum_K (\tilde{p}_h - \nabla u_h, q_h)_{0,K} = \frac{1}{3} \sum_K \Big((\Pi^1 f - \Pi^0 f)\overrightarrow{x_K x}, q_h\Big)_{0,K} \qquad \forall q_h \in (P^1(K))^2 \tag{4.10}$$

which gives (4.6).
We have proved existence and unicity of problem (4.4) and its equivalence with problem (4.5-4.6). ∎

B. Numerical analysis

We deduce from the results of the preceding section

Proposition 4.2. *If* $u \in H^4(\Omega)$,
(i) $|u - u_h|_{1,h} \le Ch^2(|u|_{3,\Omega} + |\Delta u|_{2,\Omega})$
(ii) $|u - u_h|_{0,h} \le Ch^3(|u|_{3,\Omega} + |\Delta u|_{2,\Omega})$
(iii) $\|p - p_h\|_{\operatorname{div},\Omega} \le Ch^2 |\Delta u|_{2,\Omega}$

Proof. The proof of *(i)* uses the method of proof of, e.g. [6] applied to (4.5). One has only to use the property of $P^2_{nc,0}$ to satisfy Iron's Patch test and to apply Lemma 3 of [12]. *(ii)* is proved by an Aubin-Nitsche argument as in [6]. Finally, *(iii)* results easily from the representation formula (4.6). ∎
Implementing method (4.4) is quite easy using (4.5), (4.6) and following the indications given in [15] for the implementation of the non-conforming piecewise quadratic element on triangles.

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