Convergence of a compact scheme for the pure streamfunction formulation of the unsteady Navier-Stokes system

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Abstract. This paper is devoted to the analysis of a new compact scheme for the Navier-Stokes equations in pure streamfunction formulation. Numerical results using that scheme have been reported in [2]. The scheme discussed here combines the Stephenson scheme for the biharmonic operator and ideas from box-scheme methodology. Consistency and convergence are proved for the full nonlinear system. Instead of customary periodic conditions the case of boundary conditions is addressed. It is shown that in 1D the truncation error for the biharmonic operator is $O(h^4)$ at interior points and $O(h)$ at near-boundary points. In 2D the truncation error is $O(h^2)$ at interior points (due to the cross-terms) and $O(h)$ at near-boundary points. Hence the scheme is globally of order four in the 1D periodic case and of order two in the 2D periodic case, but of order 3/2 for 1D and 2D non periodic boundary conditions. We emphasize in particular that there is no special treatment of the boundary, thus allowing robust use of the scheme. The finite element analogy of the finite difference schemes is invoked at several stages of the proofs, in order to simplify their verifications.

Keywords: Finite-difference compact schemes - Stephenson scheme - Box schemes - Finite elements - Navier-Stokes equations - Streamfunction formulation - Biharmonic problem - Fourth order problem.

1 Introduction

In a recent paper [2] we have presented a fourth-order compact scheme for the pure streamfunction formulation of the two-dimensional (incompressible) Navier-Stokes equations. We have given there a convergence analysis for the linearized model. In this paper we prove the convergence of the nonlinear scheme, without any further assumptions. Recall that the pure streamfunction formulation of the (2-D) Navier-Stokes equations is classical [14]. It has the advantage of reducing the system to a single evolution equation for the scalar streamfunction having the form

$$\frac{\partial \Delta\psi}{\partial t} + \nabla^{\perp}\psi \cdot \nabla\Delta\psi - \nu\Delta^2\psi = 0. \tag{1}$$

The velocity field is $(u,v) = \nabla^{\perp}\psi = (-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x})$, and the vorticity is $\omega = \Delta\psi$. The price to pay for the reduction of the system to a single equation is the necessity to deal with the biharmonic Δ^2 operator. There are therefore two boundary conditions imposed on ψ. For the typical "no-leak no-slip" conditions (vanishing velocity on the fixed boundary) we have

$$\nabla\psi = 0, \qquad \text{on the boundary.} \tag{2}$$

Since the function ψ is only determined up to a constant, condition (2) is equivalent to

$$\psi = \frac{\partial\psi}{\partial n} = 0, \tag{3}$$

which will be the case treated in this paper, for simplicity. Clearly (2) is equivalent to the assumption $\psi \in H_0^2$, the closure of smooth compactly supported test functions in the Sobolev space of functions having square-summable derivatives up to second order.

Our scheme can be described as follows (see [2] for details). At each time step the scheme solves a time implicit version of (1). This leads to a fourth order biharmonic problem of the form

$$\Delta\psi - \nu\Delta^2\psi = f, \tag{4}$$

subject to the boundary conditions (2).

The spatial discretization of (4) makes use of the Stephenson scheme for the the biharmonic operator introduced in [18], [11]. See also [1]. This scheme can be interpreted as a mixed scheme in $(\psi, \nabla\psi)$, similar in form to a version of a box-scheme, [13], [6]. More specifically, its design is obtained by a spline collocation procedure on a nine-point stencil, which we recall in Section 3 below.

The streamline-vorticity formulation has been extensively used for the simulation of the two-dimensional Navier-Stokes system. As representative references we mention [16], [7], [4], [8], [12] and references there. One difficult point is that "*...the $\psi - \omega$ system is inextricably coupled; BC's and solution methods must contend with this fact...*" [9, pp 431]. Indeed, one must cope with the vorticity boundary values, resulting from the fact that the relation $\Delta\psi = \omega$ is overdetermined under the condition (2). An attempt to avoid this difficulty has been made in [3], where the need to determine these values was circumvented by switching to the biharmonic equation (at each time step), exploiting the natural condition (2). The scheme presented in [2], whose convergence is proved here, has avoided all explicit mention of the vorticity by using a pure streamfunction formulation. We mention that recently in [10] a very similar algorithm has been proposed, but it deals only with the steady-state Navier-Stokes system.

The paper is organized as follows. First, we introduce in Section 2 our notation and the setup for our discrete spaces. Then we establish in Sections 3, 4 the necessary analytic properties of the scheme in 1-D and 2-D. In particular, in analogy with the coercivity of Δ^2 in H_0^2, we prove the coercivity of the discretized biharmonic operator in a suitable discrete analogue of H_0^2. We prove that the truncation error of the biharmonic scheme is of order four in 1-D, and two in 2-D, at all interior points and of first order at near-boundary points, giving a $3/2$ order of convergence rate in the natural discrete L^2 norm. Note that in the periodic case all points are interior. Then in Section 5, we prove that the same order of convergence extends to the spatial semi-discrete version of the full nonlinear scheme. We emphasize the fact that we do not need any special treatment of boundary points, and the boundary condition (2) is naturally incorporated here. As mentioned above, this causes a reduced (from 4 to 1) order of local truncation error at the boundary, and is reflected in the fact that our result yields a $3/2$ convergence rate in the discrete L^2 norm. The present convergence result can be compared to the convergence results obtained in [8], [12]. In both these papers, the time evolution is performed on the vorticity, hence a very careful treatment of the vorticity boundary conditions is required, either by "ghost-points" [8] or by replacing the condition (2) on the normal derivative of the streamfunction by boundary conditions on the vorticity ([12]) (which, as these authors observe, amounts to an algorithm for vorticity generation on the boundary).

2 Discrete spaces and basic inequalities

Let $0 \le i, j \le N$. We denote by (ih, jh) a FD mesh on the square $[0,1]^2$, with equal mesh size $h = 1/N$ in the x and y directions. We note $u_{i,j}$ a grid function on $[0,1]^2$, with $0 \le i, j \le N$. The centered and upwind derivative operators δ_x, $\delta_x^{\pm}$ are defined as usual in each direction by

$$(5) \qquad \delta_x u_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \quad , \quad \delta_x^+ u_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h} \quad , \quad \delta_x^- u_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h},$$

and similarly in the y direction:

$$(6) \qquad \delta_y u_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{h} \quad , \quad \delta_y^+ u_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{h} \quad , \quad \delta_y^- u_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{h}.$$

The centered second order derivatives are

$$(7) \qquad \delta_x^2 u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} - 2u_{i,j}}{h^2} \quad , \quad \delta_y^2 u_{i,j} = \frac{u_{i,j+1} + u_{i,j-1} - 2u_{i,j}}{h^2}.$$

The five-points Laplacian is

$$(8) \qquad \Delta_h u_{i,j} = \delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}}{h^2}.$$

The crossed derivative operators δ_{xy}^+, δ_{xy}^-, δ_{xy} are

$$(9) \qquad \delta_{xy}^+ u_{i,j} = \delta_x^+ \delta_y^+ u_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{i,j}}{h^2},$$

(10) $$\delta_{xy}^- u_{i,j} = \delta_x^- \delta_y^- u_{i,j} = \frac{u_{i,j} - u_{i,j-1} - u_{i-1,j} + u_{i-1,j-1}}{h^2},$$

(11) $$\delta_{xy} u_{i,j} = \delta_x \delta_y u_{i,j} = \frac{u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}}{4h^2}.$$

It is easy to check that

(12) $$\delta_x^2 \delta_y^2 u_{i,j} = \delta_{xy}^+ \delta_{xy}^- u_{i,j}.$$

The L_h^2 space is the space of sequences $u_{i,j}$, $0 \leq i,j \leq N$. $L_{h,0}^2$ is the subspace of $u_{i,j}$ with zero boundary conditions $u_{i,j} = 0$ for $i \in \{0, N\}$ or $j \in \{0, N\}$. The scalar product on $L_{h,0}^2$ is

(13) $$(u, v)_h = h^2 \sum_{i,j=1}^{N-1} u_{i,j} v_{i,j}.$$

with the corresponding norm

(14) $$|u|_h = \left\{ h^2 \sum_{i,j=1}^{N-1} (u_{i,j})^2 \right\}^{1/2}.$$

Furthermore, we denote by l_h^2 the space of sequences u_i, $0 \leq i \leq N$, and $l_{h,0}^2$ the subspace of sequences with zero boundary conditions. The scalar product and the norm on $l_{h,0}^2$ are

(15) $$(u, v)_h = h \sum_{i=1}^{N-1} u_i v_i \ \ , \ \ |u|_h^2 = \left\{ h \sum_{i=1}^{N-1} u_i^2 \right\}^{1/2}.$$

We also define the discrete infinity norm

(16) $$|u|_{\infty,h} = max|u_i|.$$

We skip the proof of the following lemma, which states the discrete integration by parts in $L_{h,0}^2$ for the operators $\delta_x^\pm$, δ_x^2. For each grid function $u \in L_{h,0}^2$, we denote the 1-D column vector $u^j = [u_{1,j}, u_{2,j}, ..., u_{N-1,j}]^T$, $1 \leq j \leq N-1$.

Lemma 2.1 (Discrete integration by parts) *For any $u, v \in L_{h,0}^2$, we have*

(17) $$(i) \ \ (\delta_x^+ u, v)_h = -(u, \delta_x^- v)_h.$$

(18) $$(ii) \ \ (\delta_x^2 u, v)_h = -(\delta_x^+ u, \delta_x^+ v)_h = -(\delta_x^- u, \delta_x^- v)_h.$$

Note that in (17,18), the F.D. operators are extended to the points $i = 0$, $i = N$ by

(19) $$\delta_x^\pm u_0 = \delta_x^\pm u_N = 0 \ \ , \ \ \delta_x^2 u_0 = \delta_x^2 u_N = 0.$$

Observe that this assumption is only for notational convenience, in order to have formally $\delta_x^\pm u, \delta_x^2 u \in L_{h,0}^2$. Results similar to (17, 18) in the y direction are obtained by substituting the subscript y to the subscript x. The following lemma is the counterpart of the Poincaré inequality at the discrete level

Lemma 2.2 (Discrete Poincaré inequality) *For all $u \in l_{h,0}^2$ and any $1 \leq j \leq N-1$,*

(20) $$|u^j|_h \leq 2|\delta_x^+ u^j|_h.$$

Corollary 2.1 *For all* $u \in L^2_{h,0}$

(21) $$|u|_h \le \sqrt{2}\big[|\delta_x^+ u|_h^2 + |\delta_y^+ u|_h^2\big]^{1/2}.$$

Proof:
For all $u \in l^2_{h,0}$, we have

(22) $$|u|_h^2 = h\sum_{i_0=1}^{N-1} u_{i_0}^2.$$

For all $1 \le i_0 \le N-1$,

$$\begin{aligned} u_{i_0}^2 &= \sum_{i=0}^{i_0-1}(u_{i+1}-u_i)(u_{i+1}+u_i) = \sum_{i=0}^{i_0-1} h\delta_x^+ u_i(u_{i+1}+u_i) = (\delta_x^+ u, u+Su)_h \\ &\le 2|\delta_x^+ u|_h|u|_h. \end{aligned}$$

where $(Su)_j = u_{j+1}$, $j = 0,..,N-1$. Therefore,

(23) $$|u|_h^2 = h\sum_{i_0=1}^{N-1} u_{i_0}^2 \le 2|\delta_x^+ u|_h|u|_h,$$

which gives (20).
Now for all $u \in L^2_{h,0}$, we have

(24) $$\begin{aligned} |u|_h^2 = h\sum_{j_0=1}^{N-1}|u^{j_0}|_h^2 &\le 2h\sum_{j_0=1}^{N-1}|\delta_x^+ u^{j_0}|_h|u^{j_0}|_h \\ &\le 2\Big(\sum_{j_0=1}^{N-1} h|\delta_x^+ u^{j_0}|^2\Big)^{1/2}\Big(\sum_{j_0=1}^{N-1} h|u^{j_0}|^2\Big)^{1/2} \\ &\le 2|\delta_x^+ u|_h|u|_h. \end{aligned}$$

In a similar way, we obtain in the y direction

(25) $$|u|_h^2 \le 2|\delta_y^+ u|_h|u|_h.$$

Summing (24) and (25), we obtain (21). ∎

3 The Stephenson scheme in 1-D

3.1 Design by collocation

Consider the 1-D biharmonic equation

(26) $$\begin{cases} u^{(4)}(x) = f(x) \\ u(0) = u(1) = u_x(0) = u_x(1) = 0 \end{cases}$$

Suppose that at each node $x_j = jh$, $0 \le j \le N$, of a finite difference grid, there are two unknowns u_j and $u_{x,j}$, approximating respectively $u(x_j)$ and $u_x(x_j)$, which is referred to a "mixed scheme". The values u_j, $u_{x,j}$ are solutions of the linear system, designed by the following Galerkin-collocation method. At each interior node j, $1 \le j \le N-1$, we consider a 4th order polynomial, with domain $[x_{j-1}, x_{j+1}]$

(27) $$Q(x) = a_0 + a_1(x-x_j) + a_2(x-x_j)^2 + a_3(x-x_j)^3 + a_4(x-x_j)^4.$$

The five coefficients $a_k,\ k \in \{0,1,2,3,4\}$ are defined by the five collocation conditions on the compact stencil $\{x_{j-1}, x_j, x_{j+1}\}$

$$(28)\qquad \begin{cases} Q(x_{j-1}) = u_{j-1} \quad ; \quad Q(x_j) = u_j \quad ; \quad Q(x_{j+1}) = u_{j+1} \\ Q'(x_{j-1}) = u_{x,j-1} \quad ; \quad Q'(x_{j+1}) = u_{x,j+1}. \end{cases}$$

The five coefficients of the unique polynomial (27), solution of (28), are given by

$$(29)\qquad \begin{cases} a_0 = u_j, \\ a_1 = \frac{3}{2}\delta_x u_j - \frac{1}{4}(u_{x,j+1} + u_{x,j-1}), \\ a_2 = \delta_x^2 u_j - \frac{1}{2}(\delta_x u_x)_j, \\ a_3 = \frac{1}{h^2}(\delta_x u_j - u_{x,j}) = \frac{1}{6}(\delta_x^2 u_x)_j, \\ a_4 = \frac{1}{2h^2}\left[(\delta_x u_x)_j - \delta_x^2 u_j\right]. \end{cases}$$

Now, since $Q'(x_j) = a_1$ and $Q''''(x_j) = 24a_4$, it is natural to define the following compact scheme: find $[u_0, u_1, ..., u_{N-1}, u_N]$, $[u_{x,0}, u_{x,1}, ..., u_{x,N-1}, u_{x,N}] \in l^2_{h,0}$ which solve

$$(30)\qquad \begin{cases} (P_x u_x)_j = \delta_x u_j, \quad 1 \le j \le N-1 \quad (a), \\ \delta_x^4 u_j = f(x_j) \quad 1 \le j \le N-1 \quad (b), \\ u_0 = u_1 = u_{x,0} = u_{x,N} = 0 \quad (c), \end{cases}$$

where the operators P_x, δ_x^4 are respectively defined in (31), (34).
For $u \in l^2_{h,0}$, the operator P_x is defined by

$$(31)\qquad (P_x u)_j = \frac{1}{6}u_{j-1} + \frac{2}{3}u_j + \frac{1}{6}u_{j+1} \quad ,1 \le j \le N-1.$$

P_x will be referred as the *Simpson operator* in the x direction, because the coefficients in (30) are the ones of the Simpson quadrature formula over $[x_{j-1}, x_{j+1}]$. Note also that

$$(32)\qquad P_x = I + \frac{h^2}{6}\delta_x^2.$$

We also note that the connection (30)(a) is already given in the classical book by Collatz [5, Ch. III, Eq. 2.9]. We call $\mathcal{S}$ the discrete space of grid functions $(u, u_x) \in l^2_{h,0} \times l^2_{h,0}$

$$(33)\qquad \mathcal{S} = \left\{(u, u_x) \in l^2_{h,0} \text{ such that } P_x u_x = \delta_x u\right\}.$$

In (30), we define the *Stephenson discrete biharmonic* to be the compact difference operator given on $\mathcal{S}$ by

$$(34)\qquad \delta_x^4 u_j = \frac{12}{h^2}\left\{(\delta_x u_x)_j - \delta_x^2 u_j\right\} \quad , \quad 1 \le j \le N-1.$$

This is a 1-D version of the original scheme proposed by Stephenson in [18]. Note that for simplicity, we will refer in the sequel to a grid functions in $\mathcal{S}$ by $u \in \mathcal{S}$, meaning that it is the first component of a pair $(u, u_x) \in \mathcal{S}$.
Remark: We note that the implicit scheme (30)(a) defining the grid function u_x as a function of u is exactly the one obtained in the piecewise cubic spline interpolation, see e.g. [17] . The classical question that occurs in spline interpolation about fixing the two degrees of freedom $u_{x,0}$, $u_{x,N}$ at end points is here pointless, since they are precisely given in (30)(c)

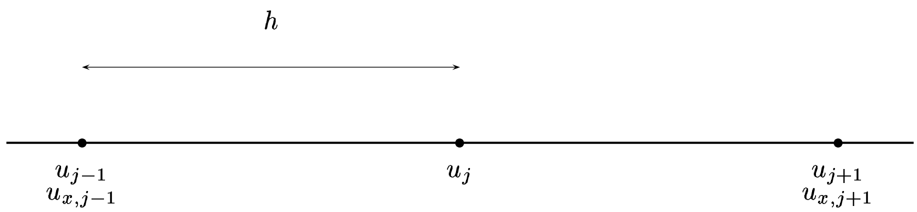

Figure 1: *Stephenson's scheme for* $u^{(4)} = f$*: The finite difference operator* $\delta_x^4 u_j$ *at point* j *is* $Q^{(4)}(x_j)$ *where* $Q(x) \in P^4[x_{j-1}, x_{j+1}]$ *is defined by the* 5 *collocated values for* u_{j-1}, u_j, u_{j+1}, $u_{x,j-1}$, $u_{x,j+1}$.

3.2 Consistency

On a periodic grid, the order of consistency can be obtained by a simple Taylor expansion at point x_j. Equivalently, one can compute the symbol of the operators. Recall that in the context of finite-difference operators, we have to use the semi-discrete Fourier transform, see e.g. [19]. In practice, if the values of the periodic grid function (u_j) are represented by $e^{ij\xi h}$, then the symbol of the linear operator L_h is $l_h(\xi)$ defined by

(35) $$L_h u_j = l_h(\xi) u_j.$$

Furthermore, if $l(\xi)$ is the symbol of L, then, the order of consistency is given by the greatest value $p > 0$ such that, (see [19]),

(36) $$l_h(\xi) - l(\xi) = O(h^p).$$

Doing so, it is quite easy to verify that the Stephenson gradient is 4th order accurate as well as the biharmonic operator (34). Indeed, we verify that
• The symbol of the discrete operator u_x in (30)(a) is

(37) $$g_h(\xi) = i\xi - \frac{1}{180} i\xi^5 h^4 + O(h^6),$$

so that the order of accuracy with respect to the operator ∂_x, whose symbol is $i\xi$ is

(38) $$g_h(\xi) - i\xi = O(h^4).$$

• The symbol of the discrete operator $\delta_x^4 u$ in (34) is

(39) $$d_h(\xi) = \xi^4 + O(h^4),$$

so that the order of accuracy with respect to ∂_x^4 is

(40) $$d_h(\xi) - (i\xi)^4 = O(h^4).$$

On a finite grid with homogeneous boundary conditions at the two ends, we have to perform a more careful analysis, because the symbolic computation no longer holds in this case.

Lemma 3.1 *Suppose that* $u(x)$ *is a regular function on* $[0, 1]$*. Then, the finite difference gradient* u_x *defined from the values* $u(x_j)$, $0 \leq j \leq N$ *by* $(P_x u_x)_j = \delta_x u(x_j)$ *has a truncation error* $(u_x)_j - u'(x_j)$ *of order 4 at each point* x_j*. More precisely,*

(41) $$|(u_x)_j - u'(x_j)| \leq C h^4 |u^{(5)}|_{\infty,[0,1]}.$$

Proof: The Stephenson gradient u_x is defined in the space $l^2_{h,0}$ by

$$(P_x u_x)_j = (\delta_x u)_j \quad , \quad 1 \le j \le N-1 \tag{42}$$

where P_x is the $N-1 \times N-1$ matrix-operator acting on $l^2_{h,0}$ as defined in (31), that is

$$P_x = \begin{pmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \dots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \ddots & \\ \vdots & & \ddots & \ddots & \\ 0 & \dots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \dots & \dots & \frac{1}{6} & \frac{2}{3} \end{pmatrix} \tag{43}$$

Consider a regular function $u(x)$, differentiable as much as needed, and denote by u', u'',..., $u^{(p)}$ its derivatives. At each point x_j, $1 \le j \le N-1$, the Taylor formula gives, (we note $u_j^{(m)} = u^{(m)}(x_j)$)

$$(\delta_x u)(x_j) = u'_j + \frac{h^2}{6} u_j^{(3)} + \frac{h^4}{2\,5!} \Big[u^{(5)}(\xi^-_{1,j}) + u^{(5)}(\xi^+_{1,j}) \Big], \tag{44}$$

where $\xi^-_{1,j} \in]x_{j-1}, x_j[$ and $\xi^+_{1,j} \in]x_j, x_{j+1}[$. Similarly, there exists $\xi^-_{2,j} \in]x_{j-1}, x_j[$, $\xi^+_{2,j} \in]x_j, x_{j+1}[$ such that

$$(\delta_x^2 u)(x_j) = u''_j + \frac{h^2}{4!} \Big[u^{(4)}(\xi^-_{2,j}) + u^{(4)}(\xi^+_{2,j}) \Big]. \tag{45}$$

We deduce that, applying (45) to u'

$$\begin{aligned} \delta_x u(x_j) - P_x u'(x_j) &= \delta_x u(x_j) - \Big[u'(x_j) + \frac{h^2}{6} \delta_x^2 u'(x_j) \Big] \\ &= u'_j + \frac{h^2}{6} u_j^{(3)} + \frac{h^4}{2.5!} \Big(u^{(5)}(\xi^-_{1,j}) + u^{(5)}(\xi^+_{1,j}) \Big) \\ &- \Big[u'_j + \frac{h^2}{6} \Big(u_j^{(3)} + \frac{h^2}{4!} [u^{(5)}(\xi^-_{2,j}) + u^{(5)}(\xi^+_{2,j})] \Big) \Big] \\ &= h^4 v_j, \end{aligned}$$

where the grid function v_j is defined by

$$v_j = \frac{1}{2.5!} \Big(u^{(5)}(\xi^+_{1,j}) + u^{(5)}(\xi^-_{1,j}) \Big) - \frac{1}{6.4!} \Big(u^{(5)}(\xi^-_{2,j}) + u^{(5)}(\xi^+_{2,j}) \Big). \tag{46}$$

Therefore, the grid function $u \in l^2_{h,0}$ verifies the identity

$$\delta_x u(x_j) - P_x u'(x_j) = h^4 v_j. \tag{47}$$

On the other hand, $u_x \in l^2_{h,0}$ is defined by

$$\delta_x u - P_x u_x = 0. \tag{48}$$

Subtracting (48) from (47), we obtain the identity in $l^2_{h,0}$

$$u' - u_x = h^4 P_x^{-1} v, \tag{49}$$

where $u' = [u'(x_1), ..., u'(x_{N-1})]$. Writing $P_x = I + \frac{h^2}{6}\delta_x^2$, the inverse of P_x is obtained by the Neumann series

$$P_x^{-1} = \sum_{k=0}^{\infty} (-\frac{h^2}{6}\delta_x^2)^k, \tag{50}$$

which gives the estimate of $|P_x^{-1}|_{\infty,h}$

(51) $$|P_x^{-1}|_{\infty,h} \le \sum_{k=0}^{\infty} \frac{h^{2k}}{6^k} |\delta_x^2|_{\infty,h}^k \le \sum_{k=0}^{\infty} (\frac{2}{3})^k = 3.$$

Observe that the matrix-operator δ_x^2 above is defined at the near boundary points $j = 1$, $j = N-1$ by

(52) $$\delta_x^2 u_1 = \frac{u_2 - 2u_1}{h^2} \quad , \quad \delta_x^2 u_{N-1} = \frac{u_{N-2} - 2u_{N-1}}{h^2}$$

We deduce now from (49) and (51) that

(53) $$|u' - u_x|_{\infty,h} \le h^4 |P_x^{-1}|_{\infty,h} |v|_{\infty,h} \le C h^4 |u^{(5)}|_{\infty,[0,1]}.$$

■

Lemma 3.2 *Suppose that $u(x)$ is a regular function on $[0,1]$. Then, the Stephenson biharmonic operator δ_x^4 defined by (34) has a truncation error $\delta_x^4 u - u^{(4)}$ of order $3/2$ in the $l^2_{h,0}$ norm*

(54) $$|\delta_x^4 u - u^{(4)}|_h \le C h^{3/2} |u^{(6)}|_{\infty,[0,1]},$$

where the notation $u^{(4)}$ stands for

(55) $$u^{(4)} = [u^{(4)}(x_1), ..., u^{(4)}(x_{N-1})] \in l^2_{h,0}.$$

Remark: The difference in accuracy between the periodic case and the non-periodic case is only due to the near boundary points 1 and $N-1$.
Proof: Recall that the finite difference biharmonic operator δ_x^4 is the 3 points compact operator, expressed in terms of u and u_x by

(56) $$\delta_x^4 u_j = \frac{12}{h^2} [\delta_x u_x - \delta_x^2 u].$$

Here, we handle the finite difference operators acting on 1-D grid functions $u = [u_1, ..., u_{N-1}]$, as $N-1 \times N-1$ matrices, see [2]. We can rewrite (30)(a) as

(57) $$P_x u_x = \frac{1}{2h} K u = \delta_x u \in l^2_{h,0},$$

where the antisymmetric matrix $K = \{K_{i,m}\}_{1 \le i,m \le N-1}$ is given by

(58) $$K_{i,m} = \begin{cases} \mathrm{sgn(m-i)}, & |m-i| = 1 \\ 0, & |m-i| \ne 1. \end{cases}$$

the operator δ_x is expressed as

(59) $$\delta_x = \frac{1}{2h} K.$$

In matrix form, (57) is simply written as

(60) $$P_x u_x = \delta_x u \text{ or } u_x = P_x^{-1} \delta_x u.$$

Using (34), the operator δ_x^4 can be rewritten in matrix form

$$\begin{aligned} \delta_x^4 &= \frac{12}{h^2} [\delta_x P_x^{-1} \delta_x - \delta_x^2] \\ &= \frac{12}{h^2} \Big[P_x^{-1} (\delta_x)^2 + [\delta_x P_x^{-1} - P_x^{-1} \delta_x] \delta_x - \delta_x^2 u \Big]. \end{aligned}$$

Applying the operator P_x, we obtain, for all $u \in l^2_{h,0}$

$$(61) \qquad P_x\Big[\delta_x^4 u - u^{(4)}\Big] = \frac{12}{h^2}\Big[(\delta_x)^2 u + [P_x\delta_x - \delta_x P_x]P_x^{-1}\delta_x u - P_x\delta_x^2 u\Big] - P_x u^{(4)} := v$$

Note that in (60-61), we refer to P_x as the symmetric positive definite matrix, (see (32-43)),

$$(62) \qquad (P_x)_{i,m} = \begin{cases} \frac{2}{3}, & m = i \\ 1/6, & |m-i| = 1 \\ 0, & |m-i| \geq 2. \end{cases}$$

Clearly the commutator $[P_x, K] = P_x K - K P_x$ is

$$(63) \qquad (P_x K - K P_x)_{i,j} = \begin{cases} -\frac{1}{3}, & i = j = 1, \\ \frac{1}{3}, & i = j = N-1, \\ 0, & \text{otherwise}, \end{cases}$$

so that the commutator $[P_x, \delta_x] = \frac{1}{2h}[P_x, K]$ is

$$(64) \qquad P_x\delta_x - \delta_x P_x = \begin{cases} -\frac{1}{6h}, & i = j = 1, \\ \frac{1}{6h}, & i = j = N-1, \\ 0, & \text{otherwise}, \end{cases}$$

This means that the operators P_x and δ_x do not commute and that the non-zero commutator values are restricted to points $j = 1$ and $j = N-1$.
Let us first evaluate (61) at points $j = 2, 3, \ldots N-2$.

$$(65) \qquad \begin{aligned} \frac{12}{h^2}\Big[(\delta_x)^2 u_j - P_x\delta_x^2 u_j\Big] - P_x u_j^{(4)} \quad &= \quad \frac{12}{h^2}\Big\{(\delta_x)^2 u_j \\ &- \quad [\frac{2}{3}\delta_x^2 u_j + \frac{1}{6}\delta_x^2 u_{j+1} + \frac{1}{6}\delta_x^2 u_{j-1}]\Big\} \\ &- \quad [\frac{2}{3}u_j^{(4)} + \frac{1}{6}u_{j-1}^{(4)} + \frac{1}{6}u_{j+1}^{(4)}]. \end{aligned}$$

The first term in the RHS of (65) is

$$(66) \qquad (\delta_x)^2 u_j = u_j'' + \frac{h^2}{3}u_j^{(4)} + \frac{32}{6!}h^4 u_j^{(6)} + \frac{128}{8!}h^6 u_j^{(8)} + Ch^8 u^{(10)}(\xi_j)$$

Using (45) for evaluating $\delta_x^2 u_m$ at $m = j-1, j, j+1$, we find that $P_x\delta_x^2 u_j$ in (65) is

$$(67) \qquad \frac{2}{3}\delta_x^2 u_j + \frac{1}{6}\delta_x^2 u_{j+1} + \frac{1}{6}\delta_x^2 u_{j-1} = u_j'' + \frac{1}{4}h^2 u_j^{(4)} + \frac{22}{6!}h^4 u_j^{(6)} + \frac{86}{8!}h^6 u_j^{(8)} + h^8 w_j$$

where $|w_j| \leq C|u^{(10)}|_{\infty,[0,1]}$. In addition, we have that the third line of the RHS in (65) is

$$(68) \qquad [\frac{2}{3}u_j^{(4)} + \frac{1}{6}u_{j-1}^{(4)} + \frac{1}{6}u_{j+1}^{(4)}] = u_j^{(4)} + \frac{1}{6}h^2 u_j^{(6)} + Ch^4 z_j$$

where $|z_j| \leq C|u^{(8)}|_{\infty,[0,1]}$. Therefore, we have, for $2 \leq j \leq N-2$

$$(69) \qquad |\frac{12}{h^2}[(\delta_x)^2 u - P_x\delta_x^2 u_j] - P_x u_j^{(4)}| \leq Ch^4|u^{(8)}|_{\infty,[0,1]}.$$

and this order is optimal. Consider now the truncation term for $j = 1$, (the computation is the same for $j = N-1$). We have

$$(70) \qquad (\delta_x^4 u)_1 = \frac{12}{h^2}[(\delta_x u_x)_1 - \delta_x^2 u_1].$$

Since $|u_{x,j} - u'_j| \le Ch^4 |u^{(5)}|_{\infty,[0,1]}$, we have

$$(71) \qquad \begin{aligned} (\delta_x u_x)_1 &= \frac{u_{x,2}}{2h} = \frac{u_{x,2} - u_{x,0}}{2h} \\ &= \frac{u'(x_2) - u'(x_0)}{2h} + \tilde{v} \\ &= u''(x_1) + \frac{h^2}{6} u^{(4)}(x_1) + \tilde{v}, \end{aligned}$$

where $\tilde{v}$ stands for a generic term such that $|\tilde{v}| \le Ch^3 |u^{(5)}|_{\infty,[0,1]}$. In addition, we have

$$(72) \qquad (\delta_x^2 u)_1 = u''(x_1) + \frac{h^2}{12} u^{(4)}(x_1) + w,$$

where

$$(73) \qquad |w| \le Ch^4 |u^{(6)}|_{\infty,[0,1]}.$$

Therefore (71), (73) show that the truncation error at the near boundary point x_1 is

$$(74) \qquad \frac{12}{h^2}\Big[(\delta_x u_x)_1 - (\delta_x^2 u)_1\Big] - u^{(4)}(x_1) = t_1, \text{ with } |t_1| \le Ch|u^{(5)}|_{\infty,[0,1]}.$$

We deduce from (61), (69), (74) that the truncation error $e = \delta_x^4 u - u^{(4)}$ is solution of the linear system

$$(75) \qquad \overline{P}_x e = v \quad , \quad v \in l^2_{h,0}, e \in l^2_{h,0},$$

where $\overline{P}_x$ is the matrix

$$(76) \qquad \overline{P}_x = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix},$$

and v is such that

$$(77) \qquad |v_1|, |v_{N-1}| \le Ch|u^{(5)}|_{\infty,[0,1]} \quad ; \quad |v_j| \le Ch^4 |u^{(8)}|_{\infty,[0,1]} \quad , \quad j = 2...N-2.$$

By the Gerschgorin theorem, $\overline{P}_x^{-1}$ is a bounded matrix independent of h, therefore $e = \overline{P}_x^{-1} v$ is such that

$$(78) \qquad |e|_h \le C|v|_h,$$

where

$$(79) \qquad |v|_h^2 \le Ch(2h^2 + \sum_{j=2}^{N-2} h^8) \le Ch^3.$$

Taking the square root in (79), we obtain (54). ∎

Remark: Note that the error at the interior points is fourth order and that the $h^{3/2}$ error is fully due to the lost of accuracy at the two boundary points $j = 1$, $j = N-1$.

3.3 Interpretation with finite elements

In this section, we establish the finite element counterpart of the scheme (30). This allows to obtain in a simple way the stability of the Stephenson finite difference operator δ_x^4. To each grid function $v \in l^2_{h,0}$, we match the function $v_h(x)$ in the finite element space $P^1_{c,0}$

(80)

$$P^1_{c,0} = \Big\{ v_h(x) \text{ such that } v_h \text{ is continuous, linear in each } [x_j, x_{j+1}], 0 \le j \le N-1, v_h(x_0) = v_h(x_N) = 0 \Big\}$$

defined by $v_h(x_j) = v_j$. Clearly, it is an isomorphism between $l^2_{h,0}$ and $P^1_{c,0}$. In addition, starting with $v \in l^2_{h,0}$, we introduce the two piecewise constant functions $\bar{v}_h$ and $v_{h,x}$ defined in each interval $K_{j+1/2} =]x_j, x_{j+1}[$ by

(81) $$\bar{v}_{h,j+1/2} = \frac{u_j + u_{j+1}}{2} \quad , \quad v_{h,x,j+1/2} = \frac{u_{j+1} - u_j}{h}.$$

An important aspect of using $P^1_{c,0}$ in the study of finite difference schemes is that it allows to streamline analytic operations like integration by parts or averaged quantities over intervals $K_{j+1/2} = [x_j, x_{j+1}]$. The $L^2[0,1]$ scalar product is denoted by

(82) $$(\varphi, \psi) = \int_0^1 \varphi(x)\psi(x)dx.$$

Writing the representation of $u_h(x)$ in $K_{j+1/2}$ as $(x_{j+1/2} = \frac{1}{2}(x_{j-1} + x_j))$.

(83) $$u_h(x)_{|K_{j+1/2}} = \bar{u}_{h,j+1/2} + u_{h,x,j+1/2}(x - x_{j+1/2}),$$

we can compare different scalar products for $(.,.)_h$ and in $L^2(0,1)$ as follows

Lemma 3.3 *For any $u, v \in l^2_{h,0}$, let $u_h(x), v_h(x) \in P^1_{c,0}$ be the corresponding finite element functions. Then we have*

(84) $$(i) \quad (u,v)_h = (u_h, v_h) + \frac{h^2}{6}(u_{h,x}, v_{h,x}) = (\bar{u}_h, \bar{v}_h) + \frac{h^2}{4}(u_{h,x}, v_{h,x}),$$

(85) $$(ii) \qquad (\delta_x u, v)_h = (u_{h,x}, v_h),$$

(86) $$(iii) \qquad (\delta_x^2 u, v)_h = -(\delta_x^+ u, \delta_x^+ v)_h = -(\delta_x^- u, \delta_x^- v)_h = -(u_{h,x}, v_{h,x}) \text{ (see (18))}$$

Proof:
The proof is an elementary computation resulting from the piecewise linearity of $u_h(x)$ in each $K_{j+1/2} = [x_j, x_{j+1}]$ given by (83). In fact, it clearly suffices to check that (84),(85),(86) hold for $u_h = \varphi_k$, $v_h = \varphi_m$, where (φ_k) is a basis of $P^1_{c,0}$. ■
Let $(u, u_x) \in \mathcal{S}$. Since $u_x \in l^2_{h,0}$, it has a matching function $p_h \in P^1_{c,0}$. On the other hand, we have the piecewise constant function $u_{h,x}$. The connection between these two functions is given by the following lemma.

Lemma 3.4 *(i) Let $u \in \mathcal{S}$ with grid gradient $u_x \in l^2_{h,0}$. Then the finite element function $p_h(x) \in P^1_{c,0}$ corresponding to u_x is the orthogonal projection of the piecewise constant function $u_{h,x}$ onto $P^1_{c,0}$. In other words, it is the unique solution $p_h \in P^1_{c,0}$ of*

(87) $$(p_h, q_h) = (u_{h,x}, q_h) \quad \forall q_h \in P^1_{c,0}.$$

In addition, we have, with $q_h \in P^1_{c,0}$ corresponding to $q \in l^2_{h,0}$

(88) $$(P_x u_x, q)_h = (p_h, q_h) = (u_x, P_x q)_h.$$

(ii) Let $u, v \in \mathcal{S}$ and $(u_h, p_h), (v_h, q_h) \in P^1_{c,0} \times P^1_{c,0}$, are the matching finite element functions, then the bilinear form $< .;. >_h$ defined on $\mathcal{S} \times \mathcal{S}$ by

$$(89) \qquad < u, v >_h = (\delta_x^4 u, v)_h = \frac{12}{h^2}\Big(u_{h,x} - p_h, v_{h,x} - q_h\Big) = (u, \delta_x^4 v)_h,$$

is a scalar product on $\mathcal{S} \times \mathcal{S}$.
(iii) Translated in terms of finite difference operators, (89) is

$$(90) \qquad \begin{aligned} < u, v >_h &= \sum_{j=0}^{N-1} h \frac{u_{x,j+1} - u_{x,j}}{h} \frac{v_{x,j+1} - v_{x,j}}{h} \\ &+ \frac{12}{h^2} \sum_{j=0}^{N-1} h \Big[\frac{u_{j+1} - u_j}{h} - \frac{1}{2}(u_{x,j} + u_{x,j+1})\Big]\Big[\frac{v_{j+1} - v_j}{h} - \frac{1}{2}(v_{x,j} + v_{x,j+1})\Big] \end{aligned}$$

Proof:
(i) The discrete gradient $u_x \in l^2_{h,0}$ is defined by

$$(91) \qquad \big[P_x u_x\big]_j = \delta_x u_j \quad , 1 \le j \le N-1,$$

where P_x is the Simpson operator given in (31). Eq. (91) is equivalent to

$$(92) \qquad (u_x, q)_h + \frac{1}{6} h^2 (\delta_x^2 u_x, q)_h = (\delta_x u, q)_h \qquad \forall q \in l^2_{h,0}.$$

Taking any $q \in l^2_{h,0}$ and the p_h corresponding to $u_x \in l^2_{h,0}$, and using (84), (85) and (86), we can rewrite (92) as

$$\begin{aligned} (u_{h,x}, q_h) &= (\delta_x u, q)_h = (u_x, q)_h + \frac{h^2}{6}(\delta_x^2 u_x, q)_h \\ &= (p_h, q_h) + \frac{h^2}{6}(p_{h,x}, q_{h,x}) - \frac{h^2}{6}(p_{h,x}, q_{h,x}) \\ &= (p_h, q_h), \end{aligned}$$

which gives (87). The symmetry of P_x is clear from the definition, see (31), (62). In addition, we have

$$(93) \qquad (P_x u_x, q)_h = (\delta_x u, q)_h = (u_{h,x}, q_h) = (p_h, q_h),$$

which proves (88).
(ii) The Stephenson biharmonic operator is, see (34),

$$(94) \qquad \delta_x^4 u_j = \frac{12}{h^2}\Big\{(\delta_x u_x)_j - \delta_x^2 u_j\Big\}.$$

We have

$$(95) \quad (\delta_x^4 u, v)_h = \frac{12}{h^2}\big[(p_{h,x}, v_h) + (u_{h,x}, v_{h,x})\big] = \frac{12}{h^2}\big[-(p_h, v_{h,x}) + (u_{h,x}, v_{h,x})\big] = \frac{12}{h^2}\Big(v_{h,x}, u_{h,x} - p_h\Big).$$

Subtracting $(q_h, u_{h,x} - p_h) = 0$ from (95), we deduce

$$(96) \qquad < u, v >_h = (\delta_x^4 u, v)_h = \frac{12}{h^2}\Big(u_{h,x} - p_h, v_{h,x} - q_h\Big).$$

We verify now that $< u, u >_h^{1/2}$ is a norm on $\mathcal{S}$. $< u, u >_h = 0$ is equivalent to $|u_{h,x} - p_h| = 0$. Therefore the piecewise affine function $p_h \in P^1_{c,0}$ is actually piecewise constant. Since it vanishes at $x = 0$ and is

continuous at any x_j, we have $p_h \equiv 0$, which is $u_{h,x} \equiv 0$. Therefore u_h is as well piecewise constant. Since $u_h(0) = 0$ we have also $u_h \equiv 0$.
Finally, we prove (90). Recall that for any $q_h \in P^1_{c,0}$, the difference $q_h - \overline{q}_h$ is orthogonal to piecewise constant functions. Thus, replacing in (96) p_h, q_h by $\overline{p}_h$, $\overline{q}_h$ respectively and noting, see (84), that

$$(97) \qquad (p_h, q_h) = (\overline{p}_h, \overline{q}_h) + \frac{h^2}{12}(p_{h,x}, q_{h,x}),$$

we get

$$(98) \qquad < u, v >_h = (p_{h,x}, q_{h,x}) + \frac{12}{h^2}(u_{h,x} - \bar{p}_h, v_{h,x} - \bar{q}_h),$$

which gives (90) using (81). ■

Remarks:
The result of Lemma (3.4)(ii) gives the uniqueness of the discrete solution of scheme (30).
The following lemma states the discrete counterpart of the equivalence of
(i) $|u_x|$ and $\|u\|_{H_1}$ for $u \in H^1_0$.
(ii) $|u_{xx}|$ and $\|u\|_{H_2}$ for $u \in H^2_0$.

Lemma 3.5 *There exist constants C, C', C'', independent of h such that for any grid function $u \in \mathcal{S}$*

$$(99) \qquad (i) \quad |u_h| \le |u|_h \le C|\delta_x^+ u|_h = C|u_{h,x}| \quad \textit{(Poincaré inequality).}$$

$$(100) \qquad (ii) \quad |\delta_x^+ u|_h \le C' < u, u >_h^{1/2}.$$

$$(101) \qquad (iii) \quad |\delta_x^+ u_x|_h \le C'' < u, u >_h^{1/2}.$$

Proof: Inequality (i) is simply the Poincaré inequality (21) in the 1-D setting, reformulated with the finite element notation. Inequality (iii) follows directly from (98) since $\delta_x^+ u_x = p_{h,x}$ as piecewise constant functions.
For (ii), we use the notation p for the grid function u_x and, as before, denote by u_h, p_h the $P^1_{c,0}$ functions associated with u, p respectively. In view of (87), we have

$$(102) \qquad \begin{aligned} |\delta_x^+ u|^2_{h'} = |u_{h,x}|^2 &= (u_{h,x} - p_h, u_{h,x} - p_h) + (p_h, p_h) \\ &= \frac{h^2}{12} < u, u >_h + |p_h|^2, \end{aligned}$$

where in the second equality we have used (96). Now, applying the Poincaré inequality (99) to p instead of u, so that,

$$(103) \qquad |p_h|^2 \le C|\delta_x^+ p|^2_h \le CC'' < u, u >_h,$$

where in the last inequality we have used (101). Inserting this inequality in (102), we obtain (100). ■

Remarks:
1- We know that $|u_{xx}|_{0,[0,1]}$ is a norm on the Sobolev space H^2_0. We may wonder if, at the discrete level, $|\delta_x^+ u_x|_h = |p_{h,x}|_{0,[0,1]}$ is a norm on $\mathcal{S}$. Actually it is a norm only if the number of points N is a even integer. We have that $p_{h,x} = 0$ implies $p_h = 0$. But the relation $P_x u_x = \delta_x u$ implies only $\delta_x u = 0$ which gives $u = 0$ only if N is an even integer.
2- For other finite difference schemes for the biharmonic problem and their link with the finite element method, we refer to the book by Li & al, [15].

3.4 Convergence of the Stephenson scheme

We derive now the following convergence result

Proposition 3.1 *Let U be the $P^1_{c,0}$ Lagrange interpolate of the exact solution $u(x)$ of (26) and $\tilde{u}$ the discrete solution of (30), then the following error estimate holds in the mesh dependent norm $<\tilde{v},\tilde{v}>_h^{1/2}$*

$$<U-\tilde{u},U-\tilde{u}>_h^{1/2}\leq Ch^{3/2}|f''|_{\infty,[0,1]}, \tag{104}$$

where the constant C is independent of h.

Proof: We estimate as usual the error by the sum of the approximation error and of the consistency error. Here, we work with the discrete norm $<.,.>_h^{1/2}$, so that there is no approximation error. We have

$$<U-\tilde{u},U-\tilde{u}>_h^{1/2}=\sup_{\tilde{v}\in\mathcal{S},\tilde{v}\neq 0}\frac{<U-\tilde{u},\tilde{v}>_h}{<\tilde{v},\tilde{v}>_h^{1/2}}. \tag{105}$$

We have

$$<U-\tilde{u},\tilde{v}>_h=(\delta_x^4(U-\tilde{u}),\tilde{v})_h=(\delta_x^4U-f,\tilde{v})_h+(f-\delta_x^4\tilde{u},\tilde{v})_h=h\sum_{j=1}^{N-1}(\delta_x^4U_j-f_j)\tilde{v}_j \tag{106}$$

Therefore, in view of Lemma 3.5,

$$\begin{aligned}|<U-\tilde{u},\tilde{v}>_h| &\leq |\delta_x^4U-f|_h|\tilde{v}|_h\\ &\leq Ch^{3/2}|\tilde{v}|_h|f''|_\infty.\end{aligned} \tag{107}$$

Using that $|\tilde{v}|_h\leq C<\tilde{v},\tilde{v}>_h^{1/2}$, see (100, 101), we find that

$$|<U-\tilde{u},\tilde{v}>_h|\leq Ch^{3/2}<\tilde{v},\tilde{v}>_h^{1/2}|f''|_{\infty,[0.1]}, \tag{108}$$

which gives the result. ■

4 The Stephenson scheme in 2D

4.1 The compact biharmonic scheme of Stephenson

We consider in this section the biharmonic problem in a square $\Omega=[0,1]^2$

$$\begin{cases}\Delta^2u(x,y)=\partial_x^4u(x,y)+\partial_y^4u(x,y)+2\partial_{xy}^2u(x,y)=f(x,y) \quad ; \quad (x,y)\in\Omega\\ u=\frac{\partial u}{\partial n}=0 \quad ; \quad \text{on } \partial\Omega\end{cases} \tag{109}$$

For any $f\in L^2(\Omega)$, the problem (109) has a unique solution $u\in H_0^2(\Omega)$. Its discrete version, using the Stephenson scheme, is to find a solution $u_{i,j}\in L^2_{h,0}$ to the equation

$$\begin{cases}\Delta_h^2u_{i,j}=f(x_i,y_j) \quad ; \quad 1\leq i,j\leq N-1\\ u_{i,j}=u_{x,i,j}=u_{y,i,j}=0 \quad ; \quad \text{for } \{i,j\}\in\{0,N\}\end{cases} \tag{110}$$

The Stephenson biharmonic operator Δ_h^2 is defined by

$$\Delta_h^2u_{i,j}=\delta_x^4u_{i,j}+\delta_y^4u_{i,j}+2\delta_x^2\delta_y^2u_{i,j}. \tag{111}$$

For any $u\in L^2_{h,0}$, the grid gradient $(u_x,u_y)\in(L^2_{h,0})^2$ is defined by

$$\begin{cases}P_xu_{x,i,j}=\delta_xu_{i,j} \quad , \quad 1\leq i,j\leq N-1\\ P_yu_{y,i,j}=\delta_yu_{i,j} \quad , \quad 1\leq i,j\leq N-1\end{cases} \tag{112}$$

where P_x, P_y are the Simpson operators, see (31),

$$
(113) \qquad \begin{cases} P_x = Id + \frac{1}{6}h^2\delta_x^2 \\ P_y = Id + \frac{1}{6}h^2\delta_y^2. \end{cases}
$$

The 1-D operators $\delta_x^4 u_{i,j}$, $\delta_y^4 u_{i,j}$ are given as functions of u, u_x, u_y by

$$
(114) \qquad \delta_x^4 u_{i,j} = \frac{12}{h^2}\Big[(\delta_x u_x)_{i,j} - (\delta_x^2 u)_{i,j}\Big] \;\; ; \;\; \delta_y^4 u_{i,j} = \frac{12}{h^2}\Big[(\delta_y u_y)_{i,j} - (\delta_y^2 u)_{i,j}\Big].
$$

For the convenience of the reader, we recall briefly how the operator Δ_h^2 has been originally derived by Stephenson, [18]. At each point (x_i, y_j) of the grid, $0 \leq i, j \leq N$, are attached the 3 unknowns $u_{i,j}$, $u_{x,i,j}$, $u_{y,i,j}$ as well as a 4th order polynomial $P_{i,j}$, simply denoted $P(x,y)$

$$
(115) \qquad P(x,y) = \sum_{x^l y^m \in \mathcal{V}_2} a_{l,m} x^l y^m,
$$

where the monomial set $\mathcal{V}$ is

$$
(116) \qquad \mathcal{V} = \{1, x, y, x^2, y^2, xy, x^3, x^2y, xy^2, y^3, x^4, x^2y^2, y^4\} \quad , \quad \#\mathcal{V} = 13.
$$

The 13 coefficients $a_{l,m}$ are uniquely determined by the following collocation conditions

(117)
- 9 collocations for $u_{l,m}$ at points (x_l, y_m) for $l \in \{i-1, i, i+1\}$, $m \in \{j-1, j, j+1\}$.
- 2 collocations for $u_{x,l,m}$ at points $(x_{i-1,j}, y_{i,j})$, $(x_{i+1,j}, y_{i,j})$.
- 2 collocations for $u_{y,l,m}$ at points $(x_{i,j}, y_{i,j+1})$, $(x_{i,j}, y_{i,j-1})$.

The collocation system gives a 13×13 linear system which can be solved explicitly. The result is given by, [18]

Lemma 4.1 *Denoting by $\Diamond$, $\Box$ and $\Diamond'$ the finite difference operators*

$$
(118) \qquad \begin{cases} \Diamond u_{i,j} = u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1}, \\ \Box u_{i,j} = u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j-1} + u_{i-1,j+1}, \\ \Diamond' u_{i,j} = u_{x,i+1,j} - u_{x,i-1,j} + u_{y,i,j+1} - u_{y,i,j-1}. \end{cases}
$$

the 13 coefficients $a_{l,m}$ of $P(x,y)$ at point (x_i, y_j) uniquely determined by the 13 conditions (117) are

$$
(119) \qquad \begin{cases} a_{0,0} = u_{i,j}, \\ a_{1,0} = \frac{3}{2}\delta_x u_{i,j} - \frac{1}{4}(u_{x,i+1,j} + u_{x,i-1,j}) \quad , \quad a_{0,1} = \frac{3}{2}\delta_y u_{i,j} - \frac{1}{4}(u_{y,i,j+1} + u_{y,i,j-1}), \\ a_{2,0} = \delta_x^2 u_{i,j} - \frac{1}{2}(\delta_x u_x)_{i,j} \quad , \quad a_{0,2} = \delta_y^2 u_{i,j} - \frac{1}{2}(\delta_y u_y)_{i,j} \quad , \quad a_{1,1} = \delta_{xy} u_{i,j}, \\ a_{3,0} = \frac{1}{6}(\delta_x^2 u_x)_{i,j} \quad , \quad a_{0,3} = \frac{1}{6}(\delta_y^2 u_y)_{i,j} \\ a_{2,1} = \frac{1}{2}(\delta_x^2 \delta_y u)_{i,j} \quad , \quad a_{1,2} = \frac{1}{2}(\delta_y^2 \delta_x u)_{i,j}, \\ a_{4,0} = \frac{1}{2h^2}\Big[(\delta_x u_x)_{i,j} - \delta_x^2 u_{i,j}\Big] \quad , \quad a_{0,4} = \frac{1}{2h^2}\Big[(\delta_y u_y)_{i,j} - \delta_y^2 u_{i,j}\Big], \\ a_{2,2} = \frac{1}{4}(\delta_x^2 \delta_y^2 u)_{i,j}. \end{cases}
$$

The gradient of $P(x,y)$ at (x_i, y_j) is $(\partial_x P(x_i, y_j), \partial_y P(x_i, y_j)) = (a_{1,0}, a_{0,1})$. Defining $u_{x,i,j} = P_x(x_i, y_j)$, $u_{y,i,j} = P_y(x_i, y_j)$, we obtain (112). Furthermore the operators δ_x^4, δ_y^4 are defined by

$$
(120) \qquad \begin{cases} \delta_x^4 u_{i,j} = \partial_x^4 P(x_i, y_j) = 24 a_{4,0}, \\ \delta_y^4 u_{i,j} = \partial_y^4 P(x_i, y_j) = 24 a_{0,4}, \end{cases}
$$

which is (114). Finally the operator $\Delta_h^2 u_{i,j}$ is defined by $\Delta_h^2 u_{i,j} = \Delta^2 P(x_i, y_j) = 24a_{4,0} + 8a_{2,2} + 24a_{0,4}$, which is (111). Furthermore, note that by expanding the finite difference operators, we find the following expression for the biharmonic operator Δ_h^2

$$\begin{aligned}\Delta_h^2 u_{i,j} &= \frac{1}{h^4}\Big\{56u_{i,j} - 16\big[u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}\big] + 2\big[u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}\big] \\ &+ 6h\big[(u_x)_{i+1,j} - (u_x)_{i-1,j} + (u_y)_{i,j+1} - (u_y)_{i,j-1}\big]\Big\}.\end{aligned}$$

4.2 Consistency and convergence for the elliptic operator

The order of consistency is deduced from the consistency in the 1-D case

Lemma 4.2 *Let u be continuously differentiable up to 6^{th} order in Ω and suppose that it vanishes, along with its gradient on $\partial\Omega$. Then, the truncation grid function $e = \Delta_h^2 u(x_i, y_j) - \Delta^2 u(x_i, y_j) \in L_{h,0}^2$ satisfies*

(121) $$|e|_h \le Ch^{3/2}\Big[|\partial u_x^6 u|_\infty + |\partial_y^6 u|_\infty + |\partial_x^4 u|_\infty + |\partial_y^4 u|_\infty\Big].$$

Proof: We have

(122) $$|\Delta_h^2 u - \Delta^2 u|_h \le |\delta_x^4 u - \partial_x^4 u|_h + |\delta_y^4 u - \partial_y^4 u|_h + 2|\delta_x^2\delta_y^2 u - \partial_x^2\partial_y^2 u|_h.$$

Using the consistency result (54) row by row and column by column we obtain

(123) $$|\delta_x^4 u - \partial_x^4 u|_h \le Ch^{3/2}|\partial_x^6 u|_{\infty,[0,1]^2},$$

(124) $$|\delta_y^4 u - \partial_y^4 u|_h \le Ch^{3/2}|\partial_y^6 u|_{\infty,[0,1]^2}.$$

The consistency for the mixed term is deduced from (45)

(125) $$|\delta_x^2\delta_y^2 u - \partial_x^2\partial_y^2 u|_h \le Ch^2\big[|\partial_x^4 u|_\infty + |\partial_y^4 u|_\infty\big].$$

■

In order to carry out convergence analysis, we need to develop discrete analogs of the basic differential estimates, as in the 1-D case of Section 3. We do this in the framework of a suitable "finite-element" space, namely, the Q_c^1 space of continuous functions in Ω satisfying the following condition: In every cell $K_{i+1/2,j+1/2} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, they are linear (separately) in x, y. Otherwise stated, it is (in every cell) in $\mathrm{Span}(1, x, y, xy)$. The subspace of interest to us is $Q_{c,0}^1$, consisting of functions (in Q_c^1) vanishing on $\partial\Omega$. It is clear how to match an element $u_h \in Q_{c,0}^1$ to a given $u \in L_{h,0}^2$; we simply take the function $a_0 + a_1 x + a_2 y + a_3 xy$ which interpolates the four values $u_{i,j}$, $u_{i+1,j}$, $u_{i,j+1}$, $u_{i+1,j+1}$. Since $u_h(x, y)$ is linear in x, (resp. in y) for every fixed value of y (resp. of x), we can in particular treat the function $u(x_i, y_j)$, for every fixed j, as a function of x_i in $l_{h,0}^2$ and then associate with it the functions u_x in $l_{h,0}^2$ (see (30)) and u_h, p_h their associated $P_{c,0}^1$ functions.
Note that these functions are determined for each fixed value of y_j. In the same way, we define the piecewise constant in $[x_j, x_{j+1}]$ function $u_{h,x}(., y_j)$. We define also the analogous functions in the y-direction. Finally, $u_{h,xy}$ is the piecewise (in cells) constant function given by the coefficient a_3 above. We now equip $Q_{c,0}^1$ with two scalar products. Each of them corresponds to an $L^2(0,1)$ product in one direction (i.e., the function is regarded as an element of $P_{c,0}^1$ in that direction), followed by an $l_{h,0}^2$ product in the other direction. They are given by

(126) $$\begin{cases} (u_h, v_h)^x = h\sum_{j=1}^{N-1}(u_h(., y_j), v_h(., y_j))_{L^2(0,1)} \\ (u_h, v_h)^y = h\sum_{i=1}^{N-1}(u_h(x_i, .), v_h(x_i, .))_{L^2(0,1)}. \end{cases}$$

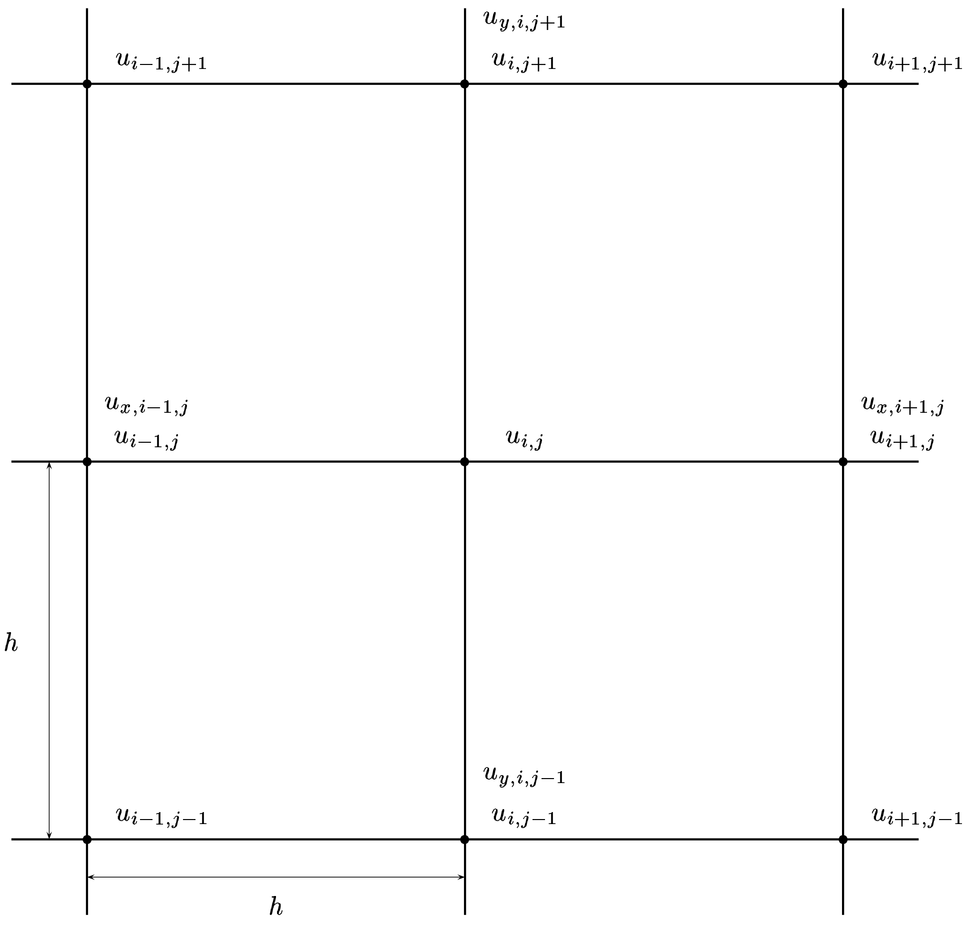

Figure 2: *Stephenson's scheme for* $\Delta^2 u = f$*: The finite difference operator* $\Delta_h^2 u_{i,j}$ *at point* (i,j) *is* $\Delta_h^2 u_{i,j} = \Delta^2 Q(x_i, y_j)$ *where* $Q(x,y) \in P^{3.5}\Big([x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]\Big)$ *is defined by the* 13 *collocated values on the picture.*

The link between the grid scalar product $(u,v)_h$ on $L^2_{h,0}$ and the two scalar products $(u_h,v_h)^x$, $(u_h,v_h)^y$ is given by (see (84)),

$$(u,v)_h = (u_h,v_h)^x + \frac{h^2}{6}(u_{h,x},v_{h,x})^x, \tag{127}$$

$$(u,v)_h = (u_h,v_h)^y + \frac{h^2}{6}(u_{h,y},v_{h,y})^y. \tag{128}$$

As in the 1-D case, see (33), we introduce here a space $\mathcal{S}$ consisting of triples $(u,u_x,u_y) \in L^2_{h,0}$, where u_x, u_y are related to u by (112). For brevity, we shall sometimes refer to the triple simply by $u \in \mathcal{S}$. As in the 1-D case, (see Lemma 3.4), we have the following result

Lemma 4.3 *Let $u \in \mathcal{S}$. Let $p_h, q_h \in Q^1_{c,0}$ correspond to u_x, u_y respectively. Then they are the projections of $u_{h,x}$, $u_{h,y}$ in the following sense*

$$(p_h,v_h)^x = (u_{h,x},v_h)^x \ , \ (q_h,v_h)^y = (u_{h,y},v_h)^y \ , \ \forall v_h \in Q^1_{c,0} \tag{129}$$

Proof: For each $1 \le j_0 \le N-1$, it results form (87) that

$$\begin{aligned}
(p_h,v_h)^x &= h\sum_{j=1}^{N-1}(p_h(.,y_j),v_h(.,y_j))_{L^2(0,1)} \\
&= h\sum_{j=1}^{N-1}(u_{h,x}(.,y_j),v_h(.,y_j))_{L^2(0,1)} \\
&= (u_{h,x},v_h)^x.
\end{aligned}$$

Therefore, the function $p_h \in Q^1_{c,0}$ matching $u_x \in L^2_{h,0}$ is the unique solution of

$$(p_h,v_h)^x = (u_{h,x},v_h)^x \quad , \quad \forall v_h \in Q^1_{c,0}. \tag{130}$$

The proof is the same for $u_{h,y}$. ■

We summarize in the following proposition the basic properties of the discrete operator Δ_h^2. As in the 1-D case, that operator gives rise to a *positive definite* bilinear form.

Proposition 4.1 *(i) Let $(u,u_x,u_y),(v,v_x,v_y) \in \mathcal{S}$, and let (u_h,p_h,q_h), (v_h,r_h,z_h) be their matches respectively in $Q^1_{c,0}$. Then, the discrete biharmonic operator Δ_h^2 defined by*

$$\Delta_h^2 u_{i,j} = \delta_x^4 u_{i,j} + \delta_y^4 u_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j} \quad , \quad 1 \le i,j \le N-1 \tag{131}$$

induces a scalar product $< u,v >_h = (\Delta_h^2 u, v)_h$ on $\mathcal{S} \times \mathcal{S}$ defined by

$$< u,v >_h = (\Delta_h^2 u,v)_h = \frac{12}{h^2}(u_{h,x}-p_h, v_{h,x}-r_h)^x + \frac{12}{h^2}(u_{h,y}-q_h, v_{h,y}-z_h)^y + 2(u_{h,xy},v_{h,xy}). \tag{132}$$

In particular, the discrete operator Δ_h^2 is symmetric-positive definite on $\mathcal{S}$.
(ii) In terms of the basic finite difference operators, the product $< u,v >_h$ is given by

$$\begin{aligned}
(\Delta_h^2 u,v)_h &= (\delta_x^+ u_x, \delta_x^+ v_x)_h + (\delta_y^+ u_y, \delta_y^+ v_y)_h + 2(\delta_x^+\delta_y^+ u, \delta_x^+\delta_y^+ v)_h \\
&+ \frac{12}{h^2}\left(\delta_x^+ u - \frac{1}{2}(u_x + u_{x,i+1,j}), \delta_x^+ v - \frac{1}{2}(v_x + v_{x,i+1,j})\right)_h \\
&+ \frac{12}{h^2}\left(\delta_y^+ v - \frac{1}{2}(u_y + u_{y,i,j+1}), \delta_y^+ v - \frac{1}{2}(v_y + v_{y,i,j+1})\right)_h.
\end{aligned} \tag{133}$$

(iii) We have the two following coercivity properties of the norm $< u, u >_h = (\Delta_h^2 u, u)_h$

$$< u, u >_h \geq C\left[|\delta_x^+ u_x|_h^2 + |\delta_y^+ u_y|_h^2 + |\delta_x^+ u_y|_h^2 + |\delta_y^+ u_x|_h^2\right], \tag{134}$$

and

$$< u, u >_h^{1/2} \geq C'|u|_h, \tag{135}$$

where C, C' are constants independent of h.

Proof: (i) By (131), we have

$$(\Delta_h^2 u, v)_h = \underbrace{(\delta_x^4 u, v)_h}_{(I)} + \underbrace{(\delta_y^4 u, v)_h}_{(II)} + 2\underbrace{(\delta_x^2 \delta_y^2 u, v)_h}_{(III)}. \tag{136}$$

We consider separately each term (I), (II), (III). For the term (I), we have

$$\begin{aligned}(\delta_x^4 u, v)_h &= h\sum_{j=1}^{N}\left(\delta_x^4 u(\cdot, y_j), v(\cdot, y_j)\right)_h \\ &= h\sum_{j=1}^{N}\left\{\frac{12}{h^2}(u_{h,x}(\cdot, y_j) - p_h, v_{h,x}(\cdot, y_j) - r_h(\cdot, y_j)\right\} \\ &= \frac{12}{h^2}(u_{h,x} - p_h, v_{h,x} - r_h)^x.\end{aligned}$$

In the same way

$$(\delta_y^4 u, v)_h = \frac{12}{h^2}(u_{h,y} - q_h, v_{h,y} - z_h)^y. \tag{137}$$

For (III), we just note that

$$(\delta_x^2 \delta_y^2 u, v)_h = (\delta_x^+ \delta_y^+ u, \delta_x^+ \delta_y^+ u)_h = (u_{h,xy}, v_{h,xy}). \tag{138}$$

Consider now the positive-definiteness of (132). Suppose that $(\Delta_h^2 u, u) = 0$, then $p_h(., y_j)$ is constant, continuous and is zero at the end points, therefore $p_h = 0$. The same result holds for q_h and u_h. We conclude that $< u, u >_h^{1/2} = (\Delta_h^2 u, u)_h^{1/2}$ is a norm in $\mathcal{S}$.
(ii) Translating (132) in term of finite difference operators, we obtain (133), as in (90).
(iii) It results from (133) that

$$(\Delta_h^2 u, u)_h \geq |\delta_x^+ u_x|_h^2 + |\delta_y^+ u_y|_h^2 + 2|\delta_x^+ \delta_y^+ u|_h^2. \tag{139}$$

For the mixed term $\delta_x^+ \delta_y^+ u$, we will show next that

$$|\delta_x^+ \delta_y^+ u|_h \geq \frac{1}{6}|\delta_x^+ u_y|_h. \tag{140}$$

Indeed

$$\delta_x^+ \delta_y^+ u_{i,j} = \frac{\delta_y^+ u_{i+1,j} - \delta_y^+ u_{i,j}}{h}. \tag{141}$$

Using $\delta_y^+ u_{i,j} = \delta_y u_{i,j} + \frac{h}{2}\delta_y^2 u_{i,j}$ and the definition of P_y, (see (113)), we deduce

$$\begin{aligned}\delta_x^+ \delta_y^+ u_{i,j} &= \frac{\delta_y u_{i+1,j} - \delta_y u_{i,j}}{h} + \frac{1}{2}\left[\delta_y^2 u_{i+1,j} - \delta_y^2 u_{i,j}\right] \\ &= \frac{1}{h}\left[u_{y,i+1,j} - u_{y,i,j}\right] + \frac{h}{6}\left[\delta_y^2 u_{y,i+1,j} - \delta_y^2 u_{y,i,j}\right] + \frac{1}{2}\left[\delta_y^2 u_{i+1,j} - \delta_y^2 u_{i,j}\right] \\ &= \delta_x^+ u_{y,i,j} + \frac{h^2}{6}\delta_y^2 \delta_x^+ u_{y,i,j} + \frac{1}{2}h\delta_y^2 \delta_x^+ u_{i,j}.\end{aligned}$$

In addition, using the definition of δ_y^2 we have

(142) $$|\delta_y^2\delta_x^+ u_y| \le \frac{4}{h^2}|\delta_x^+ u_y|_h,$$

and

(143) $$|\delta_y^2\delta_x^+ u|_h \le \frac{2}{h}|\delta_y^+\delta_x^+ u|_h.$$

Therefore, we have

$$\begin{aligned} |\delta_x^+\delta_y^+ u|_h &\ge |\delta_x^+ u_y|_h - \frac{h^2}{6}|\delta_y^2\delta_x^+ u_y|_h - \frac{h}{2}|\delta_y^2\delta_x^+ u|_h \\ &\ge |\delta_x^+ u_y|_h - \frac{2}{3}|\delta_x^+ u_y|_h - |\delta_x^+\delta_y^+ u|_h, \end{aligned}$$

which gives finally $2|\delta_x^+\delta_y^+ u|_h \ge \frac{1}{3}|\delta_x^+ u_y|_h$ or equivalently (140). We proceed in the same way in proving the symmetric estimate

(144) $$|\delta_x^+\delta_y^+ u|_h \ge \frac{1}{6}|\delta_y^+ u_x|_h.$$

Finally, the last coercivity inequality (135) is obtained starting from

(145) $$|\delta_x^+ u|_h^2 = (|u_{h,x}|^x)^2,$$

and following the same lines as in the proof of (100) in Lemma 3.5. ■
We conclude this section with the following error estimate

Proposition 4.2 *Let U be the $Q_{c,0}^1$ Lagrange interpolation of the exact solution $u(x)$ of (109) and $\tilde{u}$ the discrete solution of (110), then there exists a constant C independent of h such that*

(146) $$< U - \tilde{u}, U - \tilde{u} >^{1/2} \le Ch^{3/2}\Big(|u^{(4)}|_\infty + |u^{(6)}|_\infty\Big).$$

Proof
The proof follows the same lines as the one of Proposition 3.1. We use in particular the inequality (135).
■

5 A Stephenson based compact scheme for the streamfunction formulation of the Navier-Stokes equations

The pure streamfunction form of the Navier-Stokes equation is

(147) $$\partial_t \Delta\psi = -\nabla^\perp\psi \cdot \nabla(\Delta\psi) + \nu\Delta^2\psi.$$

The streamfunction was introduced already by Lagrange, see ([14, Ch. IV]). For simplicity, we deal only with the "no-slip" boundary condition, namely, the velocity vanishes on the boundary. This implies that we seek the streamfunction $\psi \in H_{h,0}^2$ (see [2] for a full discussion of the functional space for ψ). The notation is as follows. We denote by $\psi_{i,j} \in L_{h,0}^2$ a grid function and $\psi_{x,i,j}, \psi_{y,i,j} \in L_{h,0}^2$ the Stephenson gradient defined by

(148) $$P_x\psi_x = \delta_x\psi \quad P_y\psi_y = \delta_y\psi,$$

where the interpolation operators P_x, P_y are (see (113)),

(149) $$P_x\psi_{|i,j} = \frac{1}{6}\psi_{i-1,j} + \frac{2}{3}\psi_{i,j} + \frac{1}{6}\psi_{i+1,j} \ ; \ P_y\psi_{|i,j} = \frac{1}{6}\psi_{i,j-1} + \frac{2}{3}\psi_{i,j} + \frac{1}{6}\psi_{i,j+1},$$

The discrete gradient $\nabla_h\psi$ is defined as the pair of the discrete functions (ψ_x, ψ_y) and the discrete velocity is defined as the discrete curl of the streamfunction in the sense

(150) $$\nabla_h^{\perp}\psi_{i,j} = U_{i,j} = \big[u_{i,j}, v_{i,j}\big] = \big[-\psi_{y,i,j}, \psi_{x,i,j}\big].$$

The discrete Laplacian is defined by the standard 5 points formula

(151) $$\Delta_h\psi_{i,j} = \delta_x^2\psi_{i,j} + \delta_y^2\psi_{i,j}.$$

The discrete Stephenson biharmonic Δ_h^2 introduced in (110) is

(152) $$\Delta_h^2 u_{i,j} = \delta_x^4 u_{i,j} + \delta_y^4 u_{i,j} + 2\delta_x^2\delta_y^2 u_{i,j} \quad , \quad 1 \le i,j \le N-1$$

Δ_h^2 is a 9 points operator acting at every point (i,j) interior to the domain. The semi-discrete scheme associated with (147) consists in finding $\widetilde{\psi}(t) \in L^2_{h,0}$ which satisfies the evolution equation

(153) $$\partial_t\Delta_h\widetilde{\psi} = -\nabla_h^{\perp}\widetilde{\psi}\cdot(\Delta_h\nabla_h\widetilde{\psi}) + \nu\Delta_h^2\widetilde{\psi},$$

with initial condition

(154) $$\widetilde{\psi}_{i,j}(0) = (\psi_0)(x_i, y_j).$$

Note that in (153) and in what follows we use pointwise multiplication of functions in $L^2_{h,0}$, i.e., $(u\cdot v)_{i,j} = u_{i,j}v_{i,j}$. We denote by $e(t) = \widetilde{\psi}(t) - \psi(t)$ the difference between the computed and exact solutions. The exact solution verifies

(155) $$\partial_t\Delta_h\psi = -\nabla_h^{\perp}\psi\cdot\big[\Delta_h\nabla_h(\psi)\big] + \nu\Delta_h^2\psi + F$$

where F is the truncation error of the scheme depending on the regularity of the exact solution. We call U and $\widetilde{U}$ the discrete velocities associated to ψ, $\widetilde{\psi}$ by

(156) $$U = (-\psi_y, \psi_x) \ \ , \ \ \widetilde{U} = (-\widetilde{\psi}_y, \widetilde{\psi}_x).$$

Recall that in (156), the x and y subscripts stand for the discrete derivatives defined in (148). In particular, ψ_x, ψ_y are not the values of the exact derivatives of ψ. The error $e(t)$ evolves according to

(157) $$\partial_t\Delta e - \nu\Delta^2 e = -\big[\widetilde{U}\cdot\Delta_h(\widetilde{\psi}_x, \widetilde{\psi}_y) - U\cdot\Delta_h(\psi_x, \psi_y)\big] - F.$$

The right hand side in (157) is decomposed in 4 terms

$$\begin{aligned}\Big[(\widetilde{U}\cdot\Delta_h(\widetilde{\psi}_x, \widetilde{\psi}_y) - U\cdot\Delta_h(\psi_x, \psi_y)\Big] + F \ &= \ (\widetilde{U} - U)\cdot\Delta_h\big[(\widetilde{\psi} - \psi)_x, (\widetilde{\psi} - \psi)_y\big] \\ &+ \ (\widetilde{U} - U)\cdot\Delta_h\big[(\psi_x, \psi_y\big] \\ &+ \ U\cdot\Delta_h\big[(\widetilde{\psi} - \psi)_x, (\widetilde{\psi} - \psi)_y\big] + F.\end{aligned}$$

Taking the h scalar product with $e(t)$, we obtain

(158) $$\begin{aligned}(\partial_t\Delta e - \nu\Delta^2 e, e)_h \ &= \ -\Big((\widetilde{U} - U)\cdot\Delta_h\big[(\widetilde{\psi} - \psi)_x, (\widetilde{\psi} - \psi)_y\big], e\Big)_h \\ &- \ \big((\widetilde{U} - U)\cdot\Delta_h(\psi_x, \psi_y), e\big)_h \\ &- \ \Big(U\cdot\Delta_h\big[(\widetilde{\psi} - \psi)_x, \widetilde{\psi} - \psi)_y\big], e\Big)_h \\ &- \ (F, e)_h.\end{aligned}$$

We denote the four terms of the r.h.s. by J_1, J_2, J_3, J_4

$$\begin{aligned} J_1 &= \big((\widetilde{U}-U)\cdot\Delta_h(\widetilde{\psi}-\psi)_x,(\widetilde{\psi}-\psi)_y,e\big)_h \\ J_2 &= \big((\widetilde{U}-U)\cdot\Delta_h(\psi_x,\psi_y),e\big)_h \\ J_3 &= \big(U\cdot\Delta_h(\widetilde{\psi}-\psi)_x,\widetilde{\psi}-\psi)_y,e\big)_h \\ J_4 &= (F,e)_h. \end{aligned}$$

We estimate separately the four terms J_1, J_2, J_3, J_4.
1- **TERM J_1:**
The term J_1 is

$$J_1 = \big((\widetilde{U}-U)\cdot\Delta_h(e_x,e_y),e\big)_h. \tag{159}$$

We have

$$\widetilde{U}-U = \Big[-(\widetilde{\psi}-\psi)_y,(\widetilde{\psi}-\psi)_x\Big] = (-e_y,e_x), \tag{160}$$

where the subscripts x and y are the Stephenson derivation operators. Therefore

$$\begin{aligned} J_1 = \Big((\widetilde{U}-U)\cdot\Delta_h(e_x,e_y),e\Big)_h &= \big(-e_y(\delta_x^2 e_x+\delta_y^2 e_x)+e_x(\delta_x^2 e_y+\delta_y^2 e_y),e\big)_h \\ &= \big(-e_y(\delta_x^2 e_x+\delta_y^2 e_x),e\big)_h + \big(e_x(\delta_x^2 e_y+\delta_y^2 e_y),e\big)_h \\ &= -\big(\delta_x^2 e_x,ee_y\big)_h - \big(\delta_y^2 e_x,ee_y\big)_h + \big(\delta_x^2 e_y,ee_x\big)_h + \big(\delta_y^2 e_y,ee_x\big)_h \\ &= \big(\delta_x^+ e_x,\delta_x^+(ee_y)\big)_h + \big(\delta_y^+ e_x,\delta_y^+(ee_y)\big)_h \\ &- \big(\delta_x^+ e_y,\delta_x^+(ee_x)\big)_h - \big(\delta_y^+ e_y,\delta_y^+(ee_x)\big)_h. \end{aligned}$$

In order to formulate a discrete Leibniz rule for $w,z\in L^2_{h,0}$ we use the "shift operators" $(S_x w)_{i,j} = w_{i+1,j}$, $(S_y z)_{i,j} = z_{i,j+1}$. In terms of these operators we have

$$\delta_x^+(wz) = (S_x w)_{i,j}\delta_x^+ z + z\delta_x^+ w \tag{161}$$

which is quite easy to verify. Using (161), we expand J_1 in the sum of 8 terms

$$\begin{aligned} J_1 &= \big(\delta_x^+ e_x,(S_x e_y)_{i,j}\delta_x^+ e\big)_h + \big(\delta_x^+ e_x,e\delta_x^+ e_y\big)_h \\ &+ \big(\delta_y^+ e_x,(S_y e_y)_{i,j}\delta_y^+ e\big)_h + \big(\delta_y^+ e_x,e\delta_y^+ e_y\big)_h \\ &- \big(\delta_x^+ e_y,(S_x e_x)_{i,j}\delta_x^+ e\big)_h - \big(\delta_x^+ e_y,e\delta_x^+ e_x\big)_h \\ &- \big(\delta_y^+ e_y,(S_y e_x)_{i,j}\delta_y^+ e\big)_h - \big(\delta_y^+ e_y,e\delta_y^+ e_x\big)_h. \end{aligned}$$

There is a cancellation of terms 2 and 6 on one hand, 4 and 8 on the other hand, so that

$$J_1 = \big(\delta_x^+ e_x,(S_x e_y)\delta_x^+ e\big)_h + \big(\delta_y^+ e_x,(S_y e_y)\delta_y^+ e\big)_h + \big(\delta_x^+ e_y,(S_x e_x)\delta_x^+ e\big)_h + \big(\delta_y^+ e_y,(S_y e_x)\delta_y^+ e\big)_h. \tag{162}$$

We now observe that if $\theta\in L^2_{h,0}$, then $|\theta|_{\infty,h}\le\frac{1}{h}|\theta|_h$. We can therefore estimate J_1 as follows

$$\begin{aligned} |J_1| = \left|\Big((\widetilde{U}-U)\cdot\Delta_h(e_x,e_y),e\Big)_h\right| &\le \varepsilon\Big[|\delta_x^+ e_x|_h^2+|\delta_y^+ e_x|_h^2+|\delta_x^+ e_y|_h^2+|\delta_y^+ e_y|_h^2\Big] \\ &+ \frac{1}{4\varepsilon}\Big[|(e_x,e_y)|_{\infty,h}^2\Big(|\delta_x^+ e|_h^2+|\delta_y^+ e|_h^2\Big)\Big] \\ &\le \varepsilon\Big[|\delta_x^+ e_x|_h^2+|\delta_y^+ e_x|_h^2+|\delta_x^+ e_y|_h^2+|\delta_y^+ e_y|_h^2\Big] + \frac{C}{\varepsilon h^2}\Big[|\delta_x^+ e|_h^2+|\delta_y^+ e|_h^2\Big]^2 \end{aligned}$$

where in the last step we have used (51) to estimate $|e_x|_{\infty,h} \leq C|\delta_x^+ e|_{\infty,h}$ and $|e_y|_{\infty,h} \leq C|\delta_y^+ e|_{\infty,h}$ with a constant independent of h. The factor $\varepsilon > 0$ will be specified later.

2- Term J_2:

The term J_2 is estimated by

$$|J_2| = |\big((\widetilde{U} - U) \cdot \Delta_h(\psi_x, \psi_y), e\big)_h| \leq \frac{1}{2}C\big[|\widetilde{U} - U|_h^2 + |e|_h^2\big]. \tag{163}$$

We have used that $\Delta_h(\psi_x, \psi_y)$ is the discrete operator Δ_h composed by the Stephenson gradient applied to the exact solution, and is bounded if the exact solution is sufficiently regular. In addition, using that $\widetilde{U} - U = \big[-(\widetilde{\psi}_y - \psi_y), \widetilde{\psi}_x - \psi_x\big]$, we have

$$|\widetilde{U} - U|_h^2 = |e_x|_h^2 + |e_y|_h^2. \tag{164}$$

In addition, we have, in view of (60), (78),

$$|e_x|_h \leq C|\delta_x^+ e|_h \;\; ; \;\; |e_y|_h \leq C|\delta_y^+ e|_h, \tag{165}$$

and, due to the Poincaré inequality (21) we deduce

$$|J_2| \leq \frac{1}{2}C\Big[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\Big]. \tag{166}$$

3- TERM J_3:

We have

$$J_3 = \Big[U \cdot \Delta_h(e_x, e_y), e\Big]_h = \underbrace{(u\delta_x^2 e_x, e)_h}_{J_{3,1}} + \underbrace{(u\delta_y^2 e_x, e)_h}_{J_{3,2}} + \underbrace{(v\delta_x^2 e_y, e)_h}_{J_{3,3}} + \underbrace{(v\delta_y^2 e_y, e)_h}_{J_{3,4}}. \tag{167}$$

We have

$$J_{3,1} = (u\delta_x^2 e_x, e)_h = (\delta_x^2 e_x, ue)_h = -\Big[\delta_x^+ e_x, \delta_x^+(ue)\Big]_h. \tag{168}$$

Using (161), the term $J_{3,1}$ is estimated by

$$\begin{aligned}
|J_{3,1}| = |\big[\delta_x^+ e_x, \delta_x^+(ue)\big]_h &\leq |\delta_x^+ e_x|_h |\delta_x^+(ue)|_h \\
&\leq |\delta_x^+ e_x|_h \Big[|(S_x u)_{i,j} \delta_x^+ e|_h + |e\delta_x^+ u|_h\Big] \\
&\leq |\delta_x^+ e_x|_h \Big[|u|_{\infty,h} |\delta_x^+ e|_h + |\delta_x^+ u|_{\infty,h} |e|_h\Big].
\end{aligned}$$

Therefore, using the Poincaré inequality (21), the term $J_{3,1}$ is estimated by

$$\begin{aligned}
|J_{3,1}| &\leq \max\Big[|u|_{\infty,h}, |\delta_x^+ u|_{\infty,h}\Big]\Big[\varepsilon|\delta_x^+ e_x|_h^2 + \frac{1}{4\varepsilon}(|\delta_x^+ e|_h + |e|_h)^2\Big] \\
&\leq \max(|u|_{\infty,h}, |\delta_x^+ u|_{\infty,h})\Big[\varepsilon|\delta_x^+ e_x|_h^2 + \frac{C}{\varepsilon}(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2)\Big].
\end{aligned}$$

Using the same principle in the y direction, we obtain for the term $J_{3,2}$

$$|J_{3,2}| = |(u\delta_y^2 e_x, e)_h| \leq \max(|u|_{\infty,h}, |\delta_x^+ u|_{\infty,h})\Big[\varepsilon|\delta_y^+ e_x|_h^2 + \frac{C}{\varepsilon}(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2)\Big]. \tag{169}$$

Therefore, the estimate for the term $J_{3,1} + J_{3,2}$ is

(170)

$$|J_{3,1}+J_{3,2}| \le |J_{3,1}|+|J_{3,2}| \le \max\Big[|u|_{\infty,h}, |\delta_x^+ u|_{\infty,h}, |\delta_y^+ u|_{\infty,h}\Big]\Big[\varepsilon\Big\{|\delta_x^+ e_x|_h^2+|\delta_y^+ e_x|_h^2\Big\}+\frac{C}{\varepsilon}\Big\{|\delta_x^+ e|_h^2+|\delta_y^+ e|_h^2\Big\}\Big].$$

Treating the term $J_{3,3} + J_{3,4}$ in the same way, we obtain

(171)

$$|J_{3,3}+J_{3,4}| \le |J_{3,3}|+|J_{3,4}| \le \max\Big[|v|_{\infty,h}, |\delta_x^+ v|_{\infty,h}, |\delta_y^+ v|_{\infty,h}\Big]\Big[\varepsilon\{|\delta_x^+ e_y|_h^2+|\delta_y^+ e_y|_h^2\}+\frac{C}{\varepsilon}\{|\delta_x^+ e|_h^2+|\delta_y^+ e|_h^2\}\Big].$$

The estimate for the term J_3 is finally with $M(u) = \max\Big[|u|_\infty, |v|_\infty, |\delta_x^+ u|_\infty, |\delta_y^+ u|_\infty, |\delta_x^+ v|_\infty, |\delta_y^+ v|_\infty\Big]$,

$$|J_3| \le M(u)\Big[\varepsilon\Big\{|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_x|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_y|_h^2\Big\} + \frac{2C}{\varepsilon}\Big\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\Big\}\Big]. \tag{172}$$

4- TERM J_4:

The term J_4 is the truncation error and is of order $3/2$ (in the $|\cdot|_h$ norm) in view of Lemmas 3.1 and 4.2. For any time the term J_4 is the truncation term and is of order $3/2$. For any time $T > 0$, the term J_4 is estimated by

$$|J_4| \le C(T)|e|_h h^{3/2} \le C(T)\big[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 + h^3\big], \tag{173}$$

where $C(T)$ is a constant depending only of $T > 0$ and of the regularity of the exact solution $\psi(t)$ on $[0, T]$. We use now the following weak stability property of the Stephenson biharmonic (134) derived in section (4), which is

$$(\Delta_h^2 u, u)_h \ge C\Big[|\delta_x^+ u_x|_h^2 + |\delta_y^+ u_y|_h^2 + |\delta_x^+ u_y|_h^2 + |\delta_y^+ u_x|_h^2\Big]. \tag{174}$$

Turning back to (158), we have, on $[0, T_0]$,

$$\begin{aligned}(\frac{\partial}{\partial t}\Delta_h e, e)_h - \nu(\Delta_h^2 e, e)_h &= -\frac{1}{2}\frac{d}{dt}\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\} - \nu(\Delta_h^2 e, e)_h \\ &= -J_1 - J_2 - J_3 - J_4,\end{aligned}$$

or

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\{|\delta_x^+ e|_h^2 + \{|\delta_y^+ e|_h^2\} &= J_1 + J_2 + J_3 + J_4 - \nu(\Delta_h^2 e, e)_h \\ &\le |J_1| + |J_2| + |J_3| + |J_4| - \nu(\Delta_h^2 e, e)_h \\ &\le |J_1| + |J_2| + |J_3| + |J_4| - C\nu\big[|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_y|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_x|_h^2\big].\end{aligned}$$

Collecting the terms of the form

$$|\delta_x^+ e_x|_h^2 + |\delta_y^+ e_y|_h^2 + |\delta_x^+ e_y|_h^2 + |\delta_y^+ e_x|_h^2 \tag{175}$$

which appear in the estimates for J_1, J_2, J_3, J_4 and selecting $\varepsilon > 0$ sufficiently small, these terms are absorbed in the RHS of the last inequality. We are therefore left with the estimate

$$\frac{d}{dt}\Big\{|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\Big\} \le C\Big[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2\Big]\Big[1 + \frac{1}{h^2}(|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2)\Big] + C'h^3, \tag{176}$$

where C, C' depend on the exact solution ψ and on the viscosity coefficient ν but not on h.

In order to prove convergence of the approximate solution $\tilde{\psi}$ to the exact solution ψ using (176), we proceed as follows. We use the fact that at $t = 0$ the error $e = 0$ in order to prove an estimate for $|\delta_x^+ e|_h + |\delta_y^+ e|_h$ up to any given time $T > 0$.

Theorem 5.1 *Let $T > 0$. Then there exist constants $C, h_0 > 0$, depending possibly on T, ν and the exact solution ψ, such that, for all $0 \le t \le T$,*

(177)
$$|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \le Ch^3 \quad , \quad 0 < h \le h_0$$

Using Corollary 2.1, we obtain a 3/2 convergence rate in the discrete L^2 norm.

Proof:

Fix some $K > 0$. Observe that at $t = 0$ we have $e = 0$, hence also $\delta_x^+ e = \delta_y^+ e = 0$ (at $t = 0$). Thus, taking $h > 0$, there exists a time $\tau > 0$ (in general depending on h) such that

(178)
$$\sup_{0 \le t \le \tau} \left\{ |\delta_x^+ e|_h + |\delta_y^+ e|_h \right\} \le Kh.$$

Inserting (178) in (176) we have for $t \le \tau$

(179)
$$\frac{d}{dt}\left[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \right] \le C(1 + K^2)\left[|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \right] + C'h^3 \quad , \quad 0 < h \le h_0$$

hence by Gronwall's inequality (179) gives

(180)
$$|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \le C_1 e^{C(1+K^2)t} h^3 \quad , \quad t \le \tau$$

with a suitable constant $C_1 > 0$. Observe that in (180) τ depends on h, and define $\tau_0 = \tau_0(h)$ by

(181)
$$\tau_0 = \sup\{t > 0 \text{ such that } |\delta_x^+ e|_h + |\delta_y^+ e|_h \le Kh\}.$$

We have $\tau_0 \ge \tau$ and, as in (180), we obtain

(182)
$$|\delta_x^+ e|_h^2 + |\delta_y^+ e|_h^2 \le C_1 e^{C(1+K^2)t} h^3 \quad , \quad t \le \tau_0.$$

We can now select h_0 so small that

(183)
$$C_1 e^{C(1+K^2)T} h_0 < K.$$

Now the definition of τ_0 and (182-183) implies that, for any $0 < h \le h_0$ we have $\tau_0(h) \ge T$ and, in particular, for such h, the estimate (180) holds true for all $t \le T$. This concludes the proof of the theorem. ■

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